(4)

AD-A225 202

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)
   A critical review of the state of finite plasticity
5. TYPE OF REPORT & PERIOD COVERED
6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s)
   P. M. Naghdi
8. CONTRACT OR GRANT NUMBER(s)
   N00014-84-K-0264
9. PERFORMING ORGANIZATION NAME AND ADDRESS
   Department of Mechanical Engineering
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10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
   4324-436
11. CONTROLLING OFFICE NAME AND ADDRESS
   Mechanics Division, Solid Mechanics Program
   Office of Naval Research
   800 N. Quincy St., Arlington, VA 22217
12. REPORT DATE
   5/90
13. NUMBER OF PAGES
   80
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report)
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)
   Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
   Plasticity, finite deformation, elastic-plastic materials, areas of agreements, areas of disagreements, thermal effects, rate effects, experimental aspects, microstructural effects, crystal plasticity.
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The object of this paper is to provide a critical review of the current state of plasticity in the presence of finite deformation. In view of the controversy regarding a number of fundamental issues between several existing schools of plasticity, the areas of agreement are described separately from those of disagreement. Attention is mainly focussed on the purely mechanical, rate-independent, theory of elastic-plastic materials, although closely related topics such as rate-dependent behavior, thermal effects, experimental

and computational aspects, microstructural effects and crystal plasticity are also discussed and potentially fruitful directions are identified.

A substantial portion of this review is devoted to the area of disagreement that covers a detailed presentation of argument(s), both pro and con, for all of the basic constitutive ingredients of the rate-independent theory such as the primitive notion or definition of plastic strain, the structure of the constitutive equation for the stress response, the yield function, the loading criteria and the flow and the hardening rules. The majority of current research in finite plasticity theory, as with its infinitesimal counterpart, still utilizes a (classical) stress-based approach which inherently possesses some shortcomings for the characterization of elastic-plastic materials. These and other anomalous behavior of a stress-based formulation are contrasted with the more recent strain-based formulation of finite plasticity. A number of important features and theoretical advantages of the latter formulation, along with its computational potential and experimental interpretation, are discussed separately.

Journal of Applied Mathematics and Physics (ZAMP) 0044-2275/90/030315-80 $ 1.50 + 0.20
Vol. 41, May 1990

A critical review of the state of finite plasticity

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Contents

1. Introduction

Plasticity, as it is currently understood, embraces wide ranging fields of study which involve difficult phenomena in the mechanics of inelastic behavior of solids. Most often the current interest in the subject refers to the construction of a general theory for finitely deforming 'rate-independent' behavior of elastic-plastic materials in which viscous effects are ignored. Such a theory should, of course, be sufficiently broad enough to: (i) cover a fairly general class of materials (including metals and geomaterials), (ii) include anisotropic materials, (iii) be valid for all possible motions, and (iv) admit constitutive equations which are not too special[1]. Moreover, as is well-known, many of the basic features of the 'rate-independent' theory are shared by several closely related areas of inelastic material behavior which include time effects such as creep and relaxation, as well as thermal effects. Consequently, attention in this review is mainly focussed on the various unresolved (or not fully understood) aspects of the 'rate-independent' theory, although the nature of a number of closely related areas of the subject are also discussed.

It is fruitful to take a geometrical point of view in plasticity and this has been utilized since the early development of the subject. Thus, for example, the components of a symmetric stress tensor may be regarded as a point in a six-dimensional stress-space and the boundary of an elastic region in this space can be interpreted as a yield surface in stress space[2].

Similarly, the components of a symmetric strain tensor is a point in a six-dimensional strain space and the boundary of the elastic domain in this space is a yield surface in strain space. The equation representing the latter surface can be easily obtained from the corresponding yield surface in stress space once the stress response (such as the generalized Hooke's law) is specified. However, historically all developments on the subject relating to the classical infinitesimal deformation of elastic-plastic materials have adopted a stress-based approach and have admitted yield surfaces and associated loading criteria only in stress space. Such a stress-based approach to date remains one of the main ingredients of the majority of research in finite deformation of elastic-plastic materials, the development of which began with the work of Green and Naghdi (1965, 1966). A point of departure which employs a strain-based formulation of finite plasticity has been pursued by the present writer and co-workers since 1975. In this formulation, both the strain space and the stress space are utilized, although

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[1] While special constitutive equations are useful and of interest in a particular context or for special applications, often they are a source of anomalies in the construction of a reasonably general theory.

[2] Such geometrical interpretations correspond to the elastic range and the initial yield of the familiar one-dimensional stress-strain curv for many materials, including ductile metals.

the former is taken as *primary*. Keeping this background in mind, both the strain-space and the stress-space formulations of the subject are presented throughout this review; and their differences, along with their predictive capabilities, are discussed in considerable detail.

As is well-known, the existing formulations of a general theory of elastic-plastic material in the presence of finite deformation are somewhat controversial and there remains strong disagreements on a number of important issues between several existing schools of plasticity.[3] Because of these disagreements, the areas of agreement in this review are *separated* entirely from a large number of unresolved issues on the subject. Thus, following some preliminary remarks in section 2, areas in which there is agreement among workers in the field are placed in section 3. This is followed in section 4 by separate discussions of each of the basic ingredients which enter the 'rate-independent' theory for finitely deforming elastic-plastic materials. The content of section 4, which covers the areas of disagreement, comprises nine subsections (4A to 4I). Some of these subsections necessarily occupy more space than others; but, in each case, the aim of this review is to describe the basic nature of each unresolved issue, present the necessary argument(s) (both *pro* and *con*) in as simple a manner as possible and arrive at a definite conclusion regarding the current state of the unresolved issue in question. Some aspects of the strain-space formulation are necessarily included in section 4 as part of the comparative discussions which are the main purpose of this critical review. The remaining developments of a strain-based theory are summarized and discussed separately in section 5 and comprises three subsections (5A to 5C). A number of special features or limiting cases of the general theory are discussed in subsection 5B and include the important limiting case of the rigid-plastic materials in the presence of finite deformation and a demonstration that the well-known Prandtl-Reuss constitutive relations are strain-based even though this fact was not originally recognized. The remaining three sections of this review (sections 6–8) are devoted to several closely related areas of the subject pertaining to thermal effects and rate-dependent theory of plasticity, certain features of experimental and computational aspects so far as their bearing on the theory is concerned, as well as microstructural effects and crystal plasticity.

An effort is made to keep the number of mathematical equations to a minimum so that attention can be centered mainly on conceptual issues. When necessary, standard vector and tensor notations are used throughout

[3] This type of disagreement is always present in all difficult areas of physical sciences and should not come as a surprise. Indeed, even in the context of small deformation, there were serious disagreements during the period of approximately 1930–1960 between the proponents of the "flow" theory versus the "deformation" theory of plasticity. In this connection, see a paper by Drucker (1949) which contains a discussion of inadequacies of deformation theories of plasticity in the presence of small deformation.

this review. Generally, various entities are represented in coordinate-free forms, but it is understood that they are defined with reference to a fixed set of orthonormal basis, $\boldsymbol{e}_K$ ($K = 1, 2, 3$) say, in a three-dimensional Euclidean space.

The list of references included is far from complete or even comprehensive, but rather is intended to be representative of the existing different points of view and pertinent to the issues discussed. Since this review is mainly concerned with finite deformation of elastic-plastic materials, references to a large number of important earlier works pertaining to small elastic-plastic deformations are not included, but these and related references not explicitly cited in a particular context can, however, be easily traced from those cited here. Some of the references by the present writer and co-workers contain developments which directly bear on the content of this review. A number of statements or descriptions of physical phenomena from these earlier papers are freely quoted or paraphrased in subsequent sections.

The contents of this review article is necessarily addressed to those who have some familiarity with the subject of plasticity (at least in the context of small deformation) and who have an inkling of the nature of the existing controversies. Nevertheless, throughout this article, an effort is made to provide sufficient background so that it could be also of some use to a wider range of readers. Although all symbols are defined when first introduced, for readers' convenience a glossary of notation listing frequently used symbols is provided in a table below. It has not been possible to maintain a complete uniformity in notations and on occasion we have found it necessary to deviate from the scheme of the table and use the same symbols in different context.

A glossary of notation

Symbol	*Name or description*	*Place of definition or first occurrence*
$\boldsymbol{b}$	Body force per unit mass	(3.2)
$\mathscr{B}$	Body	Sec. 2
$\boldsymbol{D}$	Rate of deformation tensor	$(2.3)_2$
$\boldsymbol{D}_e$, $\boldsymbol{D}_p$	Measures of "elastic" and "plastic" parts of the tensor $\boldsymbol{D}$ in an additive decomposition	$(4.23)_2$
$\bar{e}\boldsymbol{I}$, $\bar{e}_p\boldsymbol{I}$	Spherical parts of total and plastic strains in the linearized theory associated with the decomposition of $(\boldsymbol{E}, \boldsymbol{E}_p)$ into their deviatoric and spherical parts	Subsection 4H (before (4.27))
$\boldsymbol{e}_K$	Orthonormal basis ($K = 1, 2, 3$) in a three-dimensional Euclidean space	Sec. 1

E	Young's modulus of elasticity	(7.5)
$\boldsymbol{E}, \dot{\boldsymbol{E}}$	Lagrangian measure of strain and its rate	(2.2), $(2.3)_1$
$\boldsymbol{E}_p, \dot{\boldsymbol{E}}_p$	Plastic strain and its rate	Subsection 4A, (4.22)
f	Yield function in stress space	(4.11)
$\hat{f}$	Inner product of the normal to the yield surface in stress space and the stress rate	$(4.17)_2$
$\boldsymbol{F}$	Deformation gradient relative to the reference position	$(2.1)_1$
$\boldsymbol{F}_e, \boldsymbol{F}_p$	Factors in the multiplicative decomposition of $\boldsymbol{F}$ associated with an intermediate stress-free configuration (see also Fig. 1)	(4.1)
g	Yield function in strain space	(4.14)
$\dot{g}$	Inner product of the normal to the yield surface in strain space and the strain rate	$(4.17)_1$
$\boldsymbol{g}_R$	The temperature gradient relative to the reference position	Subsection 6A
$\boldsymbol{I}$	Second order identity tensor	(2.2)
$\mathscr{I}$	Fourth order unit tensor	(5.10)
$\mathscr{I}_{ijkl}$	Components of unit tensor $\mathscr{I}$ referred to basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ defined similar to that following (5.11)	(5.27)
$\boldsymbol{L}$	Velocity gradient	Sec. 2
$\boldsymbol{L}_e, \boldsymbol{L}_p$	Measure of "elastic" and "plastic" parts of the tensor $\boldsymbol{L}$ in an additive decomposition	$(4.23)_1$
P	The mechanical (or stress) power	(3.6)
$\boldsymbol{P}$	The first (nonsymmetric) Piola-Kirchhoff stress tensor	(3.1)
$\boldsymbol{q}_R$	The heat flux vector measured per unit area in the reference configuration	Subsection 6A
$\boldsymbol{Q}, \bar{\boldsymbol{Q}}$	Proper orthogonal tensor-valued functions of time representing different rigid body rotations	(2.4), (4.5)
$\bar{s}\boldsymbol{I}$	Spherical part of the stress tensor in the linearized theory associated with the decomposition of $\boldsymbol{S}$ into its deviatoric and spherical parts	Subsection 4H (before (4.27))
s.r.b.m.	Abbreviation standing for superposed rigid body motions	Sec. 4 (opening paragraph)

$\boldsymbol{S}$	The second (symmetric) Piola-Kirchhoff stress tensor	(3.1)
t	Time	
$\boldsymbol{t}, {}_R\boldsymbol{t}$	The stress vectors measured, respectively, per unit area of the current and reference configurations	Sec. 3, Sec. 5
$\boldsymbol{T}$	The Cauchy stress tensor	(3.1)
$\mathscr{U}, \bar{\mathscr{U}}$	Abbreviations for the set of variables $(\boldsymbol{E}, \boldsymbol{E}_p, \kappa)$ and $(\boldsymbol{E} - \boldsymbol{E}_p, \boldsymbol{E}_p, \kappa)$	$(4.7)_{1,2}$
$\boldsymbol{v}$	Particle velocity	Sec. 2
$\mathscr{V}$	Abbreviation for the set of variables $(\boldsymbol{S}, \boldsymbol{E}_p, \kappa)$	(4.10)
$\boldsymbol{W}$	Spin (or vorticity) tensor	Subsection 4G
$\boldsymbol{W}_e, \boldsymbol{W}_p$	Measures of "elastic" and "plastic" parts of the tensor $\boldsymbol{W}$ in an additive decomposition similar to those in (4.23)	Subsection 4G
$\mathscr{W}$	Abbreviation for the set of variables $(\boldsymbol{E}_p, \kappa, \boldsymbol{\alpha})$	(5.4)
$\bar{\alpha}$	Constitutive coefficient pertaining to a measure of strain-hardening	(4.27)
$\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}$	The shift tensor (or the 'back stress') and its rate	Subsection 4H, (4.28)
γ, γ_p	Deviatoric parts of total and plastic strains in the linearized theory associated with the decomposition of $(\boldsymbol{E}, \boldsymbol{E}_p)$ into their deviatoric and spherical parts	Subsection 4H (before (4.27))
$\gamma_{ij}, \gamma_{ij}^p$	*Rectangular Cartesian components of* γ, γ_p	(5.18)
$\kappa, \dot{\kappa}$	Strain-hardening (or work-hardening) and its rate	Subsection 4B, (4.26)
μ	Shear modulus of elasticity	(4.27)
ϱ, ϱ_0	Mass densities in the reference and current configurations	(3.2), (3.3)
$\hat{\sigma}$	A symmetric second order tensor which occurs in (5.10)	(5.12)
$\boldsymbol{\tau}$	Deviatoric part of the stress tensor in the linearized theory associated with the decomposition of $\boldsymbol{S}$ into its deviatoric and spherical parts	Subsection 4H (before (4.27))
τ_{ij}	Rectangular Cartesian components of $\boldsymbol{\tau}$	(5.20)
ψ	A scalar potential, specific Helmholtz free energy	(5.30), (6.2)
Φ	A scalar function representing strain-hardening characterization as defined by (4.20)	(4.19)

2. Preliminary remarks

Prior to 1960, the developments in the rate-independent theory of elastic-plastic materials were chiefly confined to small deformation and to a large extent dealt with special material responses such as elastic-perfectly plastic or with special hardening laws such as 'isotropic' or 'kinematic' hardening[4]. This earlier state of the subject is evident from the contents of the books by Hill (1950) and by Prager and Hodge (1951), among others, as well as the coverage of several review articles, e.g., by Prager (1955), Drucker (1960), and Naghdi (1960). Nevertheless, some of the basic ideas and features of the theories with small deformation either wholly or partially can be readily extended to finitely deforming elastic-plastic materials. An account of ongoing developments in plasticity, confined mainly to infinitesimal theory, can also be found in a recent review article by Drucker (1988) which includes some material on finite plasticity and microstructure.

As noted in section 1, because of the strong disagreements on several important unresolved issues most of which also bear on conceptual characterization of the 'rate-independent' theory of plasticity, our exposition of the subject begins in section 3 with areas of agreement among the various schools of plasticity. For example, the various schools all agree on the nature of statements of the classical conservation laws and the associated invariance requirements which must hold for all theories of material behavior[5]. These include the laws of conservations of mass, linear momentum and moment of momentum.

Preparatory to the developments that follow, it is expedient to recall some basic formulas and introduce some notation and background material. Consider a body $\mathscr{B}$ with particles (material points) X and identity X with its position $\boldsymbol{X}$ in a fixed reference configuration $\boldsymbol{\kappa}_0$ in a three-dimensional Euclidean space[6] $\mathscr{E}^3$. Let $\boldsymbol{x}$ be the position vector in $\mathscr{E}^3$ occupied by X in the current configuration $\boldsymbol{\kappa}$ of $\mathscr{B}$ at time t. Also let ϱ_0 and ϱ be, respectively, the mass densities of $\mathscr{B}$ in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$. A motion of $\mathscr{B}$ is a mapping $\boldsymbol{\chi}$ defined by $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, and the deformation gradient relative to $\boldsymbol{X}$ and its determinant are:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \qquad \det \boldsymbol{F} > 0. \tag{2.1}$$

[4] An exception to the limitation of "small deformation" is the development of a special theory of rigid-plastic materials, motivated mainly by metal-forming processes, but these developments were also somewhat restrictive or were effected for rather special classes of motion or deformation.

[5] But there is a disagreement on invariance requirement associated with plastic strain (and work-hardening) which will be discussed in section 4.

[6] The reference configuration $\boldsymbol{\kappa}_0$ is not necessarily the initial configuration but for many purposes may be identified with one.

The particle velocity and the velocity gradient are denoted, respectively, by $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and $\boldsymbol{L} = \operatorname{grad} \boldsymbol{v} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, where a superposed dot denotes the material time derivative with respect to the current time t holding $\boldsymbol{X}$ fixed, $\boldsymbol{F}^{-1}$ is the inverse of $\boldsymbol{F}$ and the notation "grad" stands for the gradient operator with respect to the place $\boldsymbol{x}$ keeping t fixed. The relative (Lagrangian) symmetric strain tensor $\boldsymbol{E}$ is defined by

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{2.2}$$

where $\boldsymbol{F}^T$ stands for the transpose of $\boldsymbol{F}$ and $\boldsymbol{I}$ is the identity tensor. The rate of strain calculated from (2.2) is

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}, \qquad \boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \tag{2.3}$$

where a superposed dot denotes material time derivative and $\boldsymbol{D}$ is the rate of the deformation tensor.

Under another motion, which differs from the given one only by a superposed *rigid body motion*, the material point X moves to the place $\boldsymbol{x}^+$ in the configuration $\boldsymbol{\kappa}^+$ at time $t^+ = t + a$, where a is constant. For all quantities associated with the configuration $\boldsymbol{\kappa}^+$, we use the same symbols as those for the configuration $\boldsymbol{\kappa}$ but with an attached plus "+" sign. Thus, for the motion resulting in the configuration $\boldsymbol{\kappa}^+$, the deformation gradient, the strain and rate of strain transform according to

$$\boldsymbol{F}^+ = \boldsymbol{Q}\boldsymbol{F}, \qquad \boldsymbol{E}^+ = \boldsymbol{E}, \qquad \dot{\boldsymbol{E}}^+ = \dot{\boldsymbol{E}}, \tag{2.4}$$

while the velocity gradient and the rate of deformation tensor in $\boldsymbol{\kappa}^+$ are related to the corresponding quantities in $\boldsymbol{\kappa}$ by

$$\boldsymbol{L}^+ = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \boldsymbol{\Omega}, \qquad \boldsymbol{D}^+ = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T. \tag{2.5}$$

In (2.4)–(2.5), $\boldsymbol{Q} = \boldsymbol{Q}(t)$ is a proper orthogonal tensor-valued function of time which represents rigid body rotation and $\boldsymbol{\Omega}(t) = \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}^T(t)$ represents rigid body angular velocity. Also, the Jacobian of transformation and the mass density in the configuration $\boldsymbol{\kappa}^+$ are related to those in $\boldsymbol{\kappa}$ by

$$\det \boldsymbol{F}^+ = \det \boldsymbol{F}, \qquad \varrho^+ = \varrho. \tag{2.6}$$

Similar relationships can be calculated for other kinematical quantities but these will not be recorded here.

3. Areas of agreement

Let $\boldsymbol{T}$ be the Cauchy stress tensor (true stress), $\boldsymbol{P}$ the first (nonsymmetric) Piola-Kirchhoff stress tensor (engineering stress), $\boldsymbol{S}$ the second (symmetric) Piola-Kirchhoff stress tensor and recall the following relationships between these three stress tensors:

$$(\det \boldsymbol{F})\boldsymbol{T} = \boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T. \tag{3.1}$$

As is well-known, the distinction between the three stresses $\boldsymbol{T}$, $\boldsymbol{P}$ and $\boldsymbol{S}$ disappear in a linearized theory in which the motion and all its time and space derivatives are *small* in a sense that can be made precise, but we do not elaborate on this point here.

Setting aside the consequence of conservation of mass, we recall that the balance of moment of momentum either leads to the requirement that $\boldsymbol{T}$ be symmetric (in the Eulerian description) or imposes a similar symmetry restriction on $\boldsymbol{P}\boldsymbol{F}^T$ (in the Lagrangian description). Keeping this background in mind, we record here the differential equations of motion which follow from the balance of linear momentum. In the Eulerian description, these equations can be represented as

$$\operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{b} = \varrho \dot{\boldsymbol{v}}, \tag{3.2}$$

while in the Lagrangian description they have the form

$$\operatorname{Div} \boldsymbol{P} + \varrho_0 \boldsymbol{b} = \varrho_0 \dot{\boldsymbol{v}}, \tag{3.3}$$

where the notations "div" and "Div" stand for the divergence operator with respect to $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively.

Physical considerations demand that certain fields and functions entering the theory be *indifferent* (or *objective*)[7] to any transformation which takes the present configuration $\boldsymbol{\kappa}$ of a body rigidly into a configuration $\boldsymbol{\kappa}^+$. It is well-known that this requirement is met by the three stress tensors in (3.1) if they transform according to

$$\boldsymbol{T}^+ = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \qquad \boldsymbol{P}^+ = \boldsymbol{Q}\boldsymbol{P}, \qquad \boldsymbol{S}^+ = \boldsymbol{S}, \tag{3.4}$$

while the difference between the acceleration $\dot{\boldsymbol{v}}$ and the body force $\boldsymbol{b}$ in either of the equations of motion (3.2) or (3.3) transforms as

$$(\dot{\boldsymbol{v}} - \boldsymbol{b})^+ = \boldsymbol{Q}(\dot{\boldsymbol{v}} - \boldsymbol{b}), \tag{3.5}$$

where $\boldsymbol{Q}$ in (3.4)–(3.5) is a proper orthogonal function of time t defined following (2.5). It is important to note here that the invariance requirement $(3.4)_{1,2}$—and hence also $(3.4)_3$ and (3.5)—are deduced from a single physical requirement that the invariance properties of the stress vector $\boldsymbol{t}^+$ in the configuration $\boldsymbol{\kappa}^+$ be related to those of the stress vector $\boldsymbol{t}$ in $\boldsymbol{\kappa}$ according to the stipulation that both the magnitude of $\boldsymbol{t}$ (which acts on any boundary surface of the body with outward unit normal $\boldsymbol{n}$) and its orientation relative to $\boldsymbol{n}$ remain unchanged. For a more detailed discussion of this requirement and additional related results, see Naghdi (1972, p. 485) and Green and Naghdi (1979).

The equations of motion (3.2) or (3.3), as well as the fields which occur in these equations, are properly invariant under superposed rigid body

[7] We use the term *indifferent* (or *objective*) for brevity to mean *unaltered* or unaltered apart from orientation as defined by Green and Naghdi (1979). However, it should be emphasized that the use of the term objective here differs from the corresponding usage by some authors who appeal to the "principle of material frame-indifference" and allow $\boldsymbol{Q}$ to be any orthogonal tensor.

motions. The invariance requirements such as those in (3.4) place restrictions on the constitutive equations of any theory including, of course, those of the theory of plasticity under discussion here. Although the invariance requirements are generally accepted in continuum mechanics in the discussion of constitutive theories of material behavior, on occasion the validity of invariance requirements is questioned in the literature and some authors have stated that they could not be applicable to all materials. Response to such views will be considered in various specific contexts of plasticity in section 4. We take this opportunity, however, to draw the reader's attention to the consistency of the invariance requirements under superposed rigid body motions and the balance laws in the three-dimensional theory of continuum mechanics (Naghdi 1972, pp. 484–486) and to the additional related discussion of the invariance requirements by Green (1982) in the contexts of both continuum mechanics and molecular theory, and Woods's (1983, p. 432) acknowledgement of the importance of invariance in the latter category.

It is useful to recall here the expression for the mechanical (or stress) power P, which occurs also in the local form of balance of energy, in terms of the Lagrangian variables. Thus,

$$P = \boldsymbol{P} \cdot \operatorname{Grad} \boldsymbol{v} = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}, \tag{3.6}$$

where the notation "Grad" stands for the gradient operator with respect to $\boldsymbol{X}$.

The relation between (3.6) and the corresponding representation in terms of the Eulerian variables is simply $P = (\det \boldsymbol{F})\boldsymbol{T} \cdot \boldsymbol{D}$, in view of (3.1) and $(2.3)_1$. The latter differs by the factor $(\det \boldsymbol{F})$ from the expression $\boldsymbol{T} \cdot \boldsymbol{D}$ for the stress power in the Eulerian formulation of any theory except, of course, incompressible materials ($\det \boldsymbol{F} = 1$). It is worth noting that in the Lagrangian formulation of any theory of material behavior the constitutive equations may be specified in terms of the symmetric stress tensor $\boldsymbol{S}$ but it is the nonsymmetric stress tensor $\boldsymbol{P}$ that occurs in the equations of motion (3.3).

We now turn to some areas of plasticity for which there is a substantial degree of conceptual agreement, but will postpone the details of their mathematical representations until section 4. In the context of the rate-independent theory, the additional areas of agreement comprise:

(1) The existence of some measure of permanent deformation, e.g., plastic strain. This can be easily motivated by simple experiments when a specimen is subjected to small deformation.

(2) The idea of distinguishing between elastic and plastic regions in an elastic-plastic material. This suggests an assumption for the existence of a yield criterion (corresponding to a yield limit, for example, in a simple tension test) and more generally yield or loading surface during the plastic deformation.

(3) A constitutive equation for the stress response. This could, for example, be of the type of a generalized Hooke's law and may be motivated by unloading and reloading in a simple tension test from a state of the material beyond its purely elastic range.

(4) The necessity for adopting a constitutive equation for the rate of plastic strain, often called a *flow rule*.

(5) The introduction of at least one additional scalar variable and a corresponding constitutive equation connected with hardening behavior, often called a *hardening rule*.

4. Areas of disagreement

There is some degree of disagreements on nearly all of the main constitutive ingredients and features of plasticity in the presence of finite deformation. These features pertain to issues which arise in the phenomenological approach for characterization of the material behavior in plasticity. Some of the issues of disagreements are basic and of fundamental importance; for example, those pertaining to the definition of plastic strain and its invariance property under superposed rigid body motions (frequently abbreviated in this section as s.r.b.m.). Certain other issues while essential to the formulation of a satisfactory theory, nevertheless could be regarded as less important. In the remainder of this section, each of the issues involved is discussed separately.

4A. *Identification of plastic strain*

Although the majority of the authors of papers on plasticity introduce some measure of plastic strain, at present there is much disagreement as to exactly how this concept of plastic strain should be introduced into the theory of finitely deforming elastic-plastic materials. Part of the difficulty with the complete definition of plastic strain, denoted here by $\boldsymbol{E}_p$, arises from the fact that it involves more than just kinematical considerations. Indeed, even in the infinitesimal theory of plasticity, strain is defined in conjunction with elastic unloading during which generalized Hooke's law (or a similar constitutive equation) is assumed to hold.

Thus far, in the context of finite plasticity, two different approaches toward a satisfactory definition of plastic strain have been pursued: In one line of development investigators have attempted to define plastic strain in terms of a more primitive quantity, or in terms of a measure of strain that remains after a suitably defined unloading process; in another, plastic strain is regarded as the solution of a rate-type differential equation with the idea that the plastic strain can then be uniquely determined once appropriate

initial conditions are specified. A number of different interpretations of plastic strain have been suggested over the years, but none of these are sufficiently general to accommodate all features of permanent deformation in a finitely deforming elastic-plastic material. In fact, in the context of finite deformation, the only case so far without any ambiguity is that of finite rigid plasticity where $\boldsymbol{E} = \boldsymbol{E}_p$ [or equivalently $\boldsymbol{F} = \boldsymbol{F}_p$ using the notation of (2.1) and (2.3)]. Because of these difficulties, in their general theory of elastic-plastic materials Green and Naghdi (1965, 1966) regarded plastic strain as a primitive variable, stating certain of its properties but not defining it explicitly, and thus relegated its explicit identification to special assumptions or situations. For example, one way of identifying plastic strain is through the requirement that it be equal to the value of the total strain at zero value of stress; see the related property (3) on p. 122 of Green and Naghdi (1966)[8].

In the context of the (classical) rate-independent theory of elastic-plastic materials with small deformation, it is usual to *define* plastic strain as the difference between total strain and elastic strain, the latter being determined from the stress through generalized Hooke's law. The plastic strain at a material point is then equal to the value of the total strain when the stress is zero at that point. For a homogeneous material undergoing homogeneous deformation, the stress can be reduced to zero throughout the body by removing the applied loads and body forces; and, in this manner, the plastic strain can be readily identified. It is this notion of plastic strain that enters the usual discussion of the familiar one-dimensional tests of ductile metals.

For finite deformations, attempts have been made by several authors to define plastic strain in terms of an intermediate stress-free configuration $\bar{\kappa}$ (see Fig. 1) and the associated decomposition of the deformation gradient as a product of two tensors in the form

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p \tag{4.1}$$

and then $\boldsymbol{E}_p$ and $\boldsymbol{E}_e$ can be defined by

$$\boldsymbol{E}_p = \tfrac{1}{2}(\boldsymbol{F}_p^T \boldsymbol{F}_p - \boldsymbol{I}), \qquad \boldsymbol{E}_e = \tfrac{1}{2}(\boldsymbol{F}_e^T \boldsymbol{F}_e - \boldsymbol{I}). \tag{4.2}$$

Further, in view of $(4.2)_{1,2}$, it can be readily verified that

$$\boldsymbol{E} - \boldsymbol{E}_p = \boldsymbol{F}_p^T \boldsymbol{E}_e \boldsymbol{F}_p . \tag{4.3}$$

It should be emphasized that neither of the two factors on the right-hand side of (4.1) is necessarily the gradient of a deformation field nor satisfies any compatibility conditions.

[8] In the present context of the purely mechanical theory, this property stipulates that upon removal (not necessarily by unloading) of all stress components in the neighborhood of a point, the plastic strain $\boldsymbol{E}_p$ becomes identical to the strain tensor $\boldsymbol{E}$.

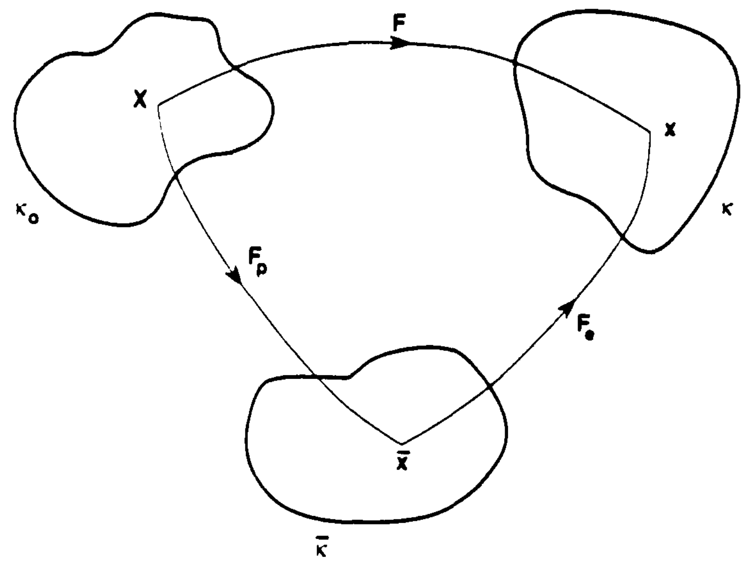

Figure 1
A schematic sketch showing the reference configuration κ_0 and the current configuration κ. Also shown is an intermediate stress-free configuration $\bar{\kappa}$ and the associated decomposition of the deformation gradient F into the factors representing the "elastic" part F_e and the "plastic" part F_p.

The form (4.1) was apparently first introduced by Kröner (1960, p. 286, Eq. (4)) with reference to linearized theory, but was subsequently utilized by Backman (1964), Lee and Liu (1967) and Lee (1969) in the context of finite deformation[9]. Issues regarding nonuniqueness and possible nonexistence of the multiplicative decomposition (4.1), as well as the matter of appropriate invariance requirements under s.r.b.m. and F_e and F_p, have been discussed by Green and Naghdi (1971) and by Casey and Naghdi (1980, 1981b).

Setting aside the nature of invariance of E_p or F_p (and F_e), we discuss now some limitations of the use of such an intermediate stress-free configuration, and the associated multiplicative decomposition, which have been raised by Green and Naghdi (1971) and also by Casey and Naghdi (1980). The most serious shortcoming of this scheme lies in the fact that the stress at a point in an elastic-plastic material can be reduced to zero without changing plastic strain only if the origin in stress space remains in the region enclosed by the yield surface. This implies a definite limitation on the usefulness of the above-mentioned definition: Indeed, except for special hardening rules (such as isotropic hardening), the yield surface may move about in stress space in a general manner as a consequence of deformation of the material. A further shortcoming is that even if the stress can be

[9] As has been remarked also by Casey (1985, p. 672), Backman used the inverse of the decomposition (4.1) in order to calculate an Eulerian strain but he then considered Lagrangian strains, while Lee and Liu added the condition that the intermediate configuration be locally stress-free.

reduced to zero at each material point, the resulting configuration will not, in general, form a configuration for the body as a whole, but only a collection of local configurations[10].

A different interpretation for the factor representing "the elastic part" of the decomposition (4.1) was proposed by Nemat-Nasser (1979); but, as Lee (1981, section 4) has argued, Nemat-Nasser's proposal is anomalous and leads to an additive decomposition of the respective gradients of the type $\boldsymbol{F}_e$ and $\boldsymbol{F}_p$ (Lee 1981, Eqs. (4.8) and (4.11)). For additional comments on this point, see Lee (1981, pp. 868–870).

The multiplicative decomposition (4.1) as used by Lee and Liu (1967) and Lee (1969) assumes that the intermediate configuration is locally stress-free and this is the reason for the restriction noted in the above penultimate paragraph. Some authors, instead of assuming an *a priori* existence of (4.1), introduce an assumption for the decomposition of the rate of deformation $\boldsymbol{D}$ in the form (4.21) of subsection 4G and then postulate a constitutive equation for $\boldsymbol{D}_p$ (see, for example, Lee 1981, Lee *et al.* 1983, and Nemat-Nasser 1982, 1983). If with reference to a particular problem this latter equation for $\boldsymbol{D}_p$ can be integrated, one can then calculate $\boldsymbol{F}_p$, along with other desired quantities, and at the same time avoid the type of restriction noted in the previous paragraph. However, such an approach also has a drawback associated with the *ad hoc* nature of the decomposition (4.21) discussed in subsection 4G.

Before ending this subsection, it may be observed once more that the notion of deformation gradient in the context of classical continuum mechanics is a purely kinematical concept. It then seems that there is no reason for expecting the same (or similar) structure as (2.2) for plastic deformation (plastic strain), which is *not* entirely a kinematical quantity; and, its identification, involves the notion of unloading from an existing elastic-plastic state. Indeed, instead of introducing $\boldsymbol{F}_p$ through (4.1), for the present it appears to be preferable to introduce the notion of plastic strain as a primitive variable represented by a symmetric second order tensor such as $\boldsymbol{E}_p$. This will avoid the restriction regarding (4.1) noted in the preceding paragraph and (at least for the present) allows a more flexible setting for the identification of $\boldsymbol{E}_p$, albeit *a posteriori*. In light of these remarks and until further progress on the nature of its identification, as in the papers of Green and Naghdi (1965, 1966) and Naghdi and Trapp (1975a) we regard plastic strain as a primitive variable represented by a second order tensor $\boldsymbol{E}_p$ and defined by its rate through an appropriate constitutive equation. In any case, such a procedure also accommodates the use of (4.1) should this be

[10] This is because the removal of loads is assumed to be performed on arbitrarily small material elements of the body. The resulting stress-free elements no longer fit together to form a continuum which can be mapped smoothly back to the present configuration of the body.

preferred by some as a starting point in the construction of a general theory as was demonstrated by Green and Naghdi (1971).

4B. *Admissibility of additional variables representing strain-hardening*

In the developments of theories of elastic-plastic materials usually one scalar, denoted here by κ, is admitted in order to represent strain-hardening (or work-hardening). In subsection 4H, however, we motivate the desirability of introducing another tensor variable, denoted by $\boldsymbol{\alpha}$, which represents the so-called shift tensor (or the 'back stress'). Should one admit more than one such scalar or tensor variables? It is a straightforward matter to include additional variables in the structure of most of the existing theories, as was noted by Green and Naghdi (1965). The inclusion of the additional variables may give a better fit with experimental data, but such additions will not alter the basic structure of the theory and no fundamentally new features will emerge. In any case, the inclusion of additional variables along with the corresponding added complexity in the constitutive equations does not seem to be fruitful at this time. With this background, for the time being we regard strain-hardening and the shift tensor as primitive variables represented by a scalar κ and a second order tensor $\boldsymbol{\alpha}$ defined by their rate through appropriate constitutive equations.

4C. *Invariance properties of variables characterizing plastic strain and work-hardening*

The invariance properties under s.r.b.m. of variables such as $\boldsymbol{E}_p$ and κ, which describe plastic deformation and the effect of work-hardening, represent a crucial issue in the construction of a general theory of elastic-plastic materials. As was noted in subsection 4A, the plastic strain $\boldsymbol{E}_p$ is not a purely kinematical variable. Nevertheless, remembering also the property $(3.4)_3$, in the construction of the Lagrangian formulation of the theory (Green and Naghdi 1965, 1966) the plastic strain $\boldsymbol{E}_p$ was assumed to have the same invariance properties as $\boldsymbol{E}$, i.e.,

$$\boldsymbol{E}_p^+ = \boldsymbol{E}_p. \tag{4.4}$$

In the same vein the effect of work-hardening characterized by the scalar κ was assumed to have the invariance property $\kappa^+ = \kappa$ under superposed rigid body motions.

Turning now to the invariance of $\boldsymbol{F}_p$ and $\boldsymbol{F}_e$, first we observe that the intermediate stress-free configuration $\bar{\kappa}$ is locally just another configuration. Then, the same line of reasoning that justifies the argument preceding (3.4) in section 3 leads us to assume that $\boldsymbol{F}_e$ and $\boldsymbol{F}_p$ should also be indifferent to a transformation (through a s.r.b.m.) that independently takes the configu-

ration $\bar{\kappa}$ to $\bar{\kappa}^+$. In this way, as in the papers of Green and Naghdi (1971) and Casey and Naghdi (1980), we are led to the transformations $F \to F^+$, $F_e \to F_e^+$ and $F_p \to F_p^+$ with

$$F^+ = F_e^+ F_p^+, \qquad F_e^+ = Q(t) F_e \bar{Q}^T(t), \qquad F_p^+ = \bar{Q}(t) F_p, \tag{4.5}$$

where F^+ is defined by $(2.4)_1$ and $\bar{Q}(t)$ is a proper orthogonal tensor-valued function of time, different from $Q(t)$ introduced previously (in section 2)[11]. It can be easily verified that E_p and E_e defined by $(4.2)_{1,2}$ will remain unaltered under the transformations (4.5) and again the work-hardening parameter κ is assumed to remain unaltered also. The reason for the appearance of two different rigid body rotation tensors in $(4.5)_2$ is simple and can be explained even without appeal to either the stress-free nature of the intermediate configuration or the elastic-plastic character of the material (Casey and Naghdi 1983a).

A number of authors, who acknowledge the existing structure of the multiplicative decomposition and its current interpretation, begin the construction of their theories with (4.1); and subsequently introduce an additional special assumption in violation of the invariance requirements for either F_e or F_p, despite the aforementioned argument. For example, Lubarda and Lee (1981) and again Lee (1981) argue that (i) it is not necessary to demand independent invariance requirements for the intermediate stress-free configuration $\bar{\kappa}$ in Fig. 1 associated with the decomposition (4.1) and that (ii) the factor F_e can be chosen to be a symmetric positive definite tensor. Both of these are faulty assumptions, as pointed out by Casey and Naghdi (1981b). With reference to the first point (i), it should be recalled that the intermediate stress-free configuration $\bar{\kappa}$, if it exists, is a possible configuration of the body; and can, in the words of Lee (1981, p. 863), "be achieved physically by ... destressing." It is precisely because of this that the configuration $\bar{\kappa}$ must be subject to *exactly the same* invariance requirements as any other possible configuration of the body, as brought out in the preceding two paragraphs. To see the fallacy of point (ii), we first note that the polar decomposition of $F_e = R_e U_e$, where the tensors R_e and U_e are proper orthogonal and symmetric positive definite, respectively, and recall the following statement from Lubarda and Lee (1981, p. 36): "For analytical convenience and with no basic loss of generality, we take the elastic deformation F_e, associated with destressing to be rotation free and hence given by U_e, a symmetric matrix." With this assumption, Lubarda and Lee choose $R_e = I$, a choice which is easily seen to violate the invariance requirements[12]. In fact, even if F_e is symmetric for a particular choice of $\bar{\kappa}$,

[11] Results of the same form as those in (4.5) were independently adopted also by Sidoroff (1970) who, however, appeals to the principle of material frame-indifference.

[12] For details, see Green and Naghdi (1971) and Casey and Naghdi (1980, 1981b).

it will not be true that F_e will be symmetric in *all* intermediate stress-free configurations that result from the first by *arbitrary* rigid displacements. Indeed, inspection of $(4.5)_2$ at once reveals that even with F_e symmetric, in general (since $\bar{Q}(t)$ is not equal to $Q(t)$) the factor F_e^+ will not be symmetric.

Similarly, in a recent paper intended to account for microstructural anisotropy in crystalline media, Dashner (1986) adopts the decomposition (4.1) but he then concludes that in the context of his analysis of finite "elastoplasticity" the intermediate stress-free configuration $\bar{\kappa}$ cannot be subjected to invariance requirements under arbitrary superposed rigid body rotation. The error in Dashner's argument has been pointed out in Casey's (1987) discussion, where it is shown that Dashner's faulty conclusion stems from his incomplete characterization of the basic kinematical ingredients required for an elastic-plastic material with microstructure.

4D. *The stress response*

By way of background, we first observe that in most developments in nonlinear continuum mechanics, for example elasticity, the deformation gradient F is well-defined through the mapping χ which carries the reference position X to the place x in κ. Then, in the context of the finite elasticity, the idea that the stress response be a function of the strain E is a meaningful one. It may be noted that geometrically the strain E may be regarded as a point in six-dimensional strain space and similarly stress tensor (either S or T) may be regarded as a point in a six-dimensional stress space.

In plasticity, however, the relevant variables are the strain E and those which describe plastic deformation specified by E_p and κ (see subsections 4A,B). Thus, for a rate-independent theory of elastic-plastic materials, the stress response may be specified by a constitutive equation of the form

$$\text{stress} = \text{function of variables } \mathscr{V} \text{ (or equivalently } \hat{\mathscr{V}}), \tag{4.6}$$

where the abbreviations $\mathscr{V}$ and $\hat{\mathscr{V}}$ stand for

$$\mathscr{V} = (E, E_p, \kappa), \qquad \hat{\mathscr{V}} = (E - E_p, E_p, \kappa). \tag{4.7}$$

The difference between the two sets of variables $(4.7)_{1,2}$ is simply in their first entry. The set $\hat{\mathscr{V}}$ is useful for certain purposes, including a derivation of the linearized theory from the nonlinear theory.

It is important to make some remarks at this point regarding the choice of $E - E_p$ in the set $(4.7)_2$. In their first paper on the subject, Green and

Naghdi made use of the set of variables of the type[13] but in the following year they also presented an alternative form r theory in terms of the variables $(4.7)_1$ and pointed out that the sions of the theory are equivalent (Green and Naghdi 1965, 1966). It should be, therefore, clear that the choice $(4.7)_2$ does not imply the additivity of elastic and plastic strains (as in the infinitesimal theory), contrary to the assertion made by Lee (1969) and similar misinterpretations by others (e.g., Simo and Ortiz 1985, p. 223). The argument refuting Lee's contention is included in a paper of Green and Naghdi (1971), where it is again emphasized that the difference $\boldsymbol{E} - \boldsymbol{E}_p$ is not an elastic strain except in a restricted or a specialized theory such as the infinitesimal theory.

Restricted forms of the constitutive equation (4.6) in which one or both of the last two entries in the arguments $\mathscr{W}$ is suppressed, e.g., a stress response of the form

$$\text{stress} = \text{function of the variable } (\boldsymbol{E} - \boldsymbol{E}_p) \tag{4.8}$$

has been utilized in many important applications. The restricted form (4.8) includes the special case of the generalized Hooke's law and is utilized in the development of the well-known Prandtl-Reuss constitutive relations for small deformation of elastic-perfectly plastic materials. In a discussion of a paper of Naghdi and Trapp (1974), which deals mainly with the development of the stress response for a class of ductile metals undergoing finite deformation, Nemat-Nasser (1974, second paragraph) has questioned the explicit inclusion of the plastic strain $\boldsymbol{E}_p$ among the arguments of $\mathscr{W}$ on the right-hand side of (4.6). To see the necessity for including $\boldsymbol{E}_p$, it will suffice to note that without the presence of this variable in $\mathscr{W}$ one could not write the right-hand side of (4.6) as a function of $\mathscr{W}$ and thus the restricted form (4.8) and hence also the Prandtl-Reuss equations could not be recovered.

For later use, we now assume the invertibility of the stress constitutive equation so that for fixed values of $\boldsymbol{E}_p$ and κ, (4.6) in terms of $\mathscr{W}$ may be inverted to yield an expression for $\boldsymbol{E}$ in the form

$$\boldsymbol{E} = \hat{\boldsymbol{E}}(\mathscr{V}), \tag{4.9}$$

where the abbreviation $\mathscr{V}$ stands for

$$\mathscr{V} = (\boldsymbol{S}, \boldsymbol{E}_p, \kappa) \tag{4.10}$$

[13] Recall that the theory developed in the two papers of Green and Naghdi (1965, 1966) is thermodynamical. It was developed in the context of the stress-space formulation and by its specialization to the isothermal case may be identified with the corresponding purely mechanical theory under discussion. The difference measure $\boldsymbol{E} - \boldsymbol{E}_p$ was denoted by $\boldsymbol{E}'$ in Green and Naghdi (1965), while the Helmholtz free energy (and hence the stress) was assumed to depend on the set of variables of the type $(4.7)_2$. Further, it was stated in the 1965 paper that (i) in general $\boldsymbol{E}'$ depends on the stress and $\boldsymbol{E}_p$ (apart from any dependence on temperature) and that (ii) alternatively the stress may be taken to be a function of the variables of the type $(4.7)_1$; see the footnotes on pages 265 and 268 of the 1965 paper.

and where in writing (4.10) and in anticipation of certain later conclusions we have chosen $\boldsymbol{S}$ to represent the stress. Alternatively, an inverted form such as (4.9) could be displayed in terms of the Cauchy stress tensor $\boldsymbol{T}$ with the use of (3.1), but this would involve dependence on the deformation gradient $\boldsymbol{F}$ in addition to the set $\mathscr{V}$.

With a view toward the clarification of an important issue in the construction of finite plasticity theory, it is necessary to comment here in regard to the choice of the stress tensor in the constitutive equation for the stress response. In this connection, we recall that interpretations of the physical processes associated with elastic and elastic-plastic materials have been advanced in the paper of Palgen and Drucker (1983, section 2). Despite this, it is difficult to understand their (Palgen and Drucker 1983) preference for the Eulerian description in terms of the Cauchy stress tensor $\boldsymbol{T}$ especially since the transformation between $\boldsymbol{T}$ and the (Lagrangian) symmetric stress $\boldsymbol{S}$ simply requires the use of (3.1). Similar preferential status has been indicated or assumed by others, for example by Onat (see his discussion included at the end of the paper of Green and Naghdi 1966, p. 131) and by Lee (1969). In addition to his preference for the Cauchy stress, Lee (1969) seems to insist that the stress response must be a function of only the elastic strain $\boldsymbol{E}_e$ defined by $(4.2)_2$. As already noted earlier in this subsection, such a special assumption is useful in a particular application but it is much too restrictive in the development of a general theory.

4E. *Yield criteria*

Historically, within the framework of what is now referred to as the stress-space formulation of plasticity, the notion of yield condition was first introduced in the work of Tresca (1867, 1878)[14] and utilized by Saint-Venant (1870) and Lévy (1870) in their development of a theory of rigid-perfectly plastic solid. Another well-known yield condition is that of von Mises (1913) and was utilized in the development of a system of equations for rigid-perfectly plastic materials (known as St. Venant-Lévy-Mises equations), as well as in the Prandtl-Reuss constitutive relations for an elastic-perfectly plastic material undergoing small deformation.

Subsequently, within the scope of the theory of elastic-plastic materials with small deformation, the notion of yield in the stress-space formulation was generalized to cover work-hardening materials, i.e., the existence of a yield function, say f, which depends on the stress, as well as on plastic strain and a work-hardening parameter. Such a scalar-valued yield (or loading)

[14] These two papers, which are in English, represent a survey of Tresca's own work during the period 1864-1872. Additional related background on Tresca's work and that of other investigators can be found in Bell's (1973) monograph.

function, which may be expressed in terms of the variables (4.10), is such that for fixed values of $(\boldsymbol{E}_p, \kappa)$ the equation

$$f(\mathscr{V}) = 0 \tag{4.11}$$

represents a closed orientable hypersurface $\partial\mathscr{S}$ of dimension five, enclosing an open region $\mathscr{S}$ in stress space[15]. The function f is chosen such that $f(\mathscr{V}) < 0$ for all points in $\mathscr{S}$, and the hypersurface $\partial\mathscr{S}$ is called the yield (or loading) surface in stress space (see Fig. 3).

Once a constitutive equation of the type (4.6) is adopted for $\boldsymbol{S}$, i.e.,

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\mathscr{U}), \tag{4.12}$$

then it is always possible to construct a yield function g in strain space[16]. Thus, from the left-hand side of (4.11) and using also (4.12), an expression for g can be found through the formula

$$f(\mathscr{V}) = f(\hat{\boldsymbol{S}}(\mathscr{U}), \boldsymbol{E}_p, \kappa) = g(\mathscr{U}). \tag{4.13}$$

Conversely, a constitutive relation of the type (4.9) can be used to obtain f from g. Again for fixed values of $(\boldsymbol{E}_p, \kappa)$, the equation

$$g(\mathscr{U}) = 0 \tag{4.14}$$

represents a hypersurface $\partial\mathscr{E}$ which encloses an open region $\mathscr{E}$ in strain space (see Fig. 2) and has the same geometrical properties as the yield surface $\partial\mathscr{S}$ in stress space. In particular, for later use we note the following relationship between the normals $\partial g/\partial \boldsymbol{E}$ and $\partial f/\partial \boldsymbol{S}$:

$$\frac{\partial g}{\partial \boldsymbol{E}} = \mathscr{L}^T\left[\frac{\partial f}{\partial \boldsymbol{S}}\right], \tag{4.15}$$

where

$$\mathscr{L} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{E}}. \tag{4.16}$$

It is clear from the identity (4.13) that a point in stress space belongs to the elastic region $\mathscr{S}$ if and only if the corresponding point in strain space belongs to $\mathscr{E}$. Likewise, a point in stress space lies on the yield surface $\partial\mathscr{S}$ if and only if the corresponding point in strain space lies on $\partial\mathscr{E}$. In view of these properties, we may refer to g as the yield (or loading) function in strain space, to $\partial\mathscr{E}$ as the yield (or loading) surface in strain space and to $\mathscr{E}$ as the elastic region in strain space.

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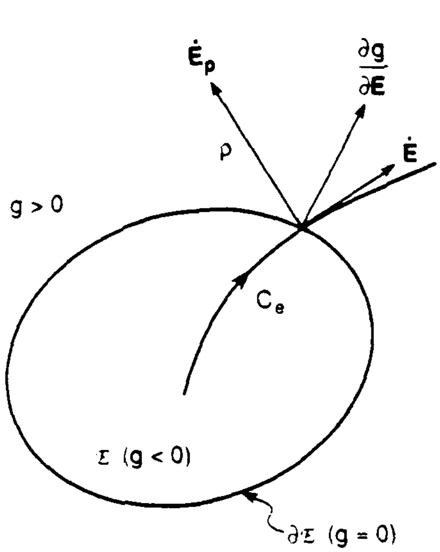
[15] Although we have elected to write here the condition in terms of the tensor $\boldsymbol{S}$, such a yield condition can alternatively be written in terms of the Cauchy stress tensor $\boldsymbol{T}$ with the use of (3.1), but the result will involve dependence on the deformation gradient $\boldsymbol{F}$ in addition to the set $\mathscr{V}$.

[16] This is a point of departure in the strain-space formulation to which reference was also made in section 1. The advantages of this formulation becomes clear in the remaining subsections of section 4 and the first two subsections of section 5.



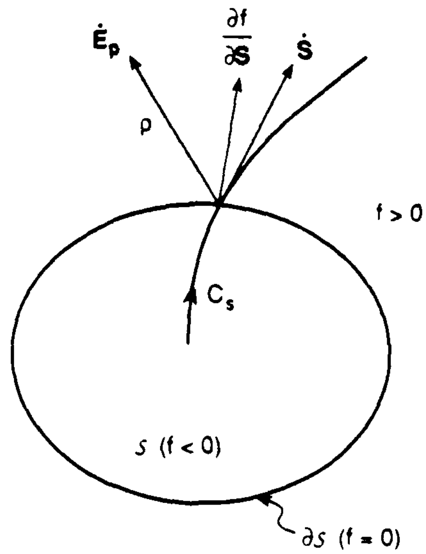


Figure 2
A sketch of the yield surface $\partial\mathscr{E}$ and the elastic region $\mathscr{E}$ in six-dimensional strain space. Also shown are a strain trajectory C_e, its tangent vector $\dot{E}$, the normal vector $\partial g/\partial E$ to $\partial\mathscr{E}$, as well as vectors representing ρ and the rate of plastic strain $\dot{E}_p$. During loading the yield surface $\partial\mathscr{E}$ is pushed outwards by the strain trajectory C_e.

Figure 3
A sketch of the yield surface $\partial\mathscr{S}$ and the elastic region $\mathscr{S}$ in six-dimensional stress space. Also shown are a stress trajectory C_s, its tangent vector $\dot{S}$, the normal vector $\partial f/\partial S$ to $\partial\mathscr{S}$, as well as vectors representing ρ and the rate of plastic strain $\dot{E}_p$.

It follows from the remarks in the preceding paragraph, that it is equally acceptable to introduce the notion of yield condition (and yield function) first with reference to strain space and then calculate the corresponding yield function in stress space by a formula of the type (4.13). There is no basic reason for limiting the consideration of yield to only the stress space. Both stress *and* strain variables must occur in any theory of plasticity, as is obvious also from a plot of typical uniaxial stress-strain curve, so both stress space and strain space should be considered. In the early development of the theory for rigid-plastic solids, the natural ingredients of the idealized "rigid-perfectly plastic" *model* necessarily required that the yield condition be stated only in terms of stress; and, historically speaking, this seems to be the main reason for the consideration of yield in stress space alone for almost a century.

In the usual development of plasticity theory, a single loading function is admitted for characterization of plastic flow throughout the entire history of deformation in the manner described in the preceding paragraphs of this subsection. However, in recent years a number of investigators have attempted to provide improvements in the theory and its predictability by the

introduction of additional yield (or loading) surfaces. The notion of a family of loading surfaces, distinct from the yield surface (i.e., the boundary of a region in stress space) within which both unloading and reloading result in elastic strains only, was evidently first introduced in a paper of Phillips and Sierakowski (1965). Such multi-loading (or multi-yield) surfaces were further discussed by Mroz (1967) and adopted by others[17] for the purpose of describing special phenomena such as hardening characterization, cyclic stress-strain response which exhibit saturation hardening, etc. While the idea of introducing multi-loading surfaces may have had some merit at the time of its inception, at least as a measure of expediency for certain applications, its role in any acceptable general theory appears to be unnecessary. In fact, in several recent calculations and comparisons of theoretical predictions with experimental results for cyclic loading and related phenomena, the use of multi-loading surfaces (Eisenberg 1976, Dafalias and Popov 1976 and others) has not shown any improved capability or predictability over and above those resulting with the use of a single loading surface[18].

Again, in recent years some investigators appear to have placed special emphasis either on the possibility of developing a theory of plasticity without introducing a "yield surface" or on requiring that the theory be capable of predicting the existence of a "yield surface" (Owen 1970, Tokuoka 1971), rather than postulating the existence of a yield or loading surface *ab initio*. As has been pointed out by Green and Naghdi (1973), such endeavors at present appear to be somewhat illusory in the following sense: whichever way the theory is developed, it must necessarily involve some assumptions that ultimately result in a surface separating an elastic region from an elastic-plastic one; and it appears to be largely a matter of taste as to which kind of assumptions are preferred at the outset of the development of the theory.

The nature of yield criteria has been discussed so far in terms of smooth yield or loading functions in the sense that a unique normal exists at each point of the hypersurfaces $\partial\mathcal{S}$ and $\partial\mathcal{E}$, as indicated also in Figs. 2 and 3. In the rest of this subsection, we briefly comment on the nature of singular yield or loading surfaces. An example of such yield surfaces is, of course, the yield condition of Tresca which has corners in principal stress space[19]. In the context of small deformation, Koiter (1953a) was the first to recognize the potential advantage of using such piecewise linear yield functions; and, with the use of Tresca's yield condition, obtained an analytical solution for

[17] For example, the papers of Eisenberg and Phillips (1971), Krieg (1975), Eisenberg (1976), Dafalias and Popov (1976), among others.

[18] See the results of Caulk and Naghdi (1978) and Naghdi and Nikkel (1984, 1986) who have obtained their results with the use of a single loading surface.

[19] The corresponding yield surface in principal strain space, which can be obtained with the use of the linearized version of (4.8), will also have corners.

partially plastic thick-walled tubes which includes the effect of elastic compressibility[20]. Further discussions of a number of features of singular yield conditions can be found in the papers of Koiter (1953b), Prager (1953) and Hodge (1956a,b).

The existing state of the knowledge on singular yield surfaces and their features are all limited to infinitesimal plasticity, but the corresponding development in the context of finite plasticity is worth exploring. In fact, it is natural to expect that the use of piecewise linear yield functions within the scope of finite plasticity is likely to lead to a fruitful endeavor in obtaining both analytical (or at least partially analytical) and numerical results.

4F. *Primacy of strain-space formulation. Loading criteria*

The basic ingredients of any finite theory of elastic-plastic materials that admits a yield (or loading) function in a stress-space setting involve two features which are generalizations of corresponding ideas in the classical infinitesimal plasticity for work-hardening materials. These two features are: (a) the rate of plastic strain (and the rate of work-hardening) depend linearly on the rate of stress; and (b) loading criteria in terms of a scalar $\hat{f}$, which represents the inner product of the rate of stress and the normal to the yield surface (4.11) in stress space. Such a stress-space formulation in terms of the variables (4.10) leads to unreliable results in regions of material behavior such as those that correspond to the flat and falling portions of the typical engineering stress-strain curve for uniaxial tension of ductile metals (see Fig. 1 of Naghdi and Trapp 1975a). Moreover, the stress-space formulation does not reduce directly to the theory of elastic-perfectly plastic materials even for infinitesimal deformation[21]. In order to overcome these difficulties, Naghdi and Trapp (1975a) proposed an alternative strain-space formulation whose main features are: (a′) the rate of plastic strain (and also the rate of work-hardening) depend on the rate of strain; and (b′) loading criteria in terms of a scalar $\hat{g}$, which represents the inner product of the rate of strain and the normal to the yield surface (4.13) in strain space[22].

It is clear from the discussion of the previous subsection that a unique yield surface in strain space can be constructed from a given yield surface in stress space; and, moreover, the normal to the yield surface in strain space

[20] An analytical solution of this type with the use of Prandtl-Reuss constitutive relations (which employs the von Mises yield condition), for example for a thick-walled cylinder in the state of plane strain, prior to Koiter's (1953a) paper was available only for the incompressible case. For further background information prior to Koiter's paper on this problem, see Prager and Hodge (1951, pp 100-109).

[21] Recall that the condition for *loading* in the stress-space formulation of the elastic-perfectly plastic case is not the same as the corresponding condition for work-hardening materials.

[22] As will become evident below, the strain-space formulation of plasticity renders the theory valid for the full range of elastic-plastic deformation and includes the theory of elastic-perfectly plastic materials as a special case.

can be calculated from the normal to the yield surface in stress space. Given this background, we take the loading function (4.14) in strain space as primary and preliminary to the discussion of the loading criteria define the quantities $\hat{g}$ and $\hat{f}$ (introduced in the preceding paragraph) by

$$\hat{g} = \frac{\partial g}{\partial \boldsymbol{E}} \cdot \dot{\boldsymbol{E}}, \qquad \hat{f} = \frac{\partial f}{\partial \boldsymbol{S}} \cdot \dot{\boldsymbol{S}}. \tag{4.17}$$

Corresponding to the motion which results in the configuration $\boldsymbol{\kappa}$, we associate with each particle of the continuum a smooth oriented curve C_e which lies in strain space and is parameterized by time. The curve C_e may be referred to as a strain trajectory. In the present discussion, the strain trajectories are restricted to lie initially in $\mathscr{E}$ or on $\partial\mathscr{E}$. When evaluated on the yield surface $\partial\mathscr{E}$, $(4.15)_1$ can be interpreted as the inner product between the (six-dimensional) tangent vector to a strain trajectory and the outward unit normal to $\partial\mathscr{E}$ (assuming that $\partial g/\partial \boldsymbol{E}$ is not zero). Then, the loading criteria of the strain-space formulation are defined to be:

$$\begin{aligned}
&\text{(a) } g < 0 && \text{(elastic state)},\\
&\text{(b) } g = 0 \text{ and } \hat{g} < 0 && \text{(unloading from an elastic-plastic state)},\\
&\text{(c) } g = 0 \text{ and } \hat{g} = 0 && \text{(neutral loading)},\\
&\text{(d) } g = 0 \text{ and } \hat{g} > 0 && \text{(loading)}.
\end{aligned} \tag{4.18}$$

In light of the criteria (4.16), in an elastic state the strain trajectory C_e lies in $\mathscr{E}$; during unloading, C_e intersects the yield surface $\partial\mathscr{E}$ and is directed inwards; during neutral loading, C_e is tangent to $\partial\mathscr{E}$; and during loading, C_e intersects $\partial\mathscr{E}$ and is directed outwards. Figure 2 illustrates various quantities associated with the strain-space formulation discussed in this subsection.

For a given motion χ (introduced in section 2) and associated strain trajectory C_e, with the use of the constitutive equations (4.6), we may obtain the corresponding stress trajectory C_s, a continuous oriented curve in stress space; see, in this connection, Fig. 3 which illustrates various quantities in the theory associated with the stress space. Once the yield condition and the loading criteria of the strain-space formulation are adopted as primary, $\hat{f}$ can be calculated from $(4.17)_2$ and the loading conditions (not criteria) in stress space can be deduced from those of the strain space, but the former conditions in stress space are not equivalent to the latter and the conditions involving $\hat{f}$ alone cannot be used as loading criteria. As pointed out by Naghdi and Trapp (1975a), the induced conditions in stress space which accompany the strain space loading criteria are $g(\mathscr{U}) = f(\mathscr{V})$ by (4.13), as well as $\hat{g} = \hat{f}$ associated with (4.18a,b,c). The latter conditions imply, respectively, $f < 0$ (elastic state), $f = 0$, $\hat{f} < 0$ (during unloading) and $f = 0$, $\hat{f} = 0$ (during neutral loading). Also, the interpretation that may be associated with $f = 0$, $\hat{f} > 0$ will become clear presently.

The loading conditions in stress space that can exist in conjunction with (4.18) suggest a natural classification of strain-hardening into three distinct types—hardening, softening and perfectly plastic. It turns out that algebraically the three categories are distinguished, respectively, by positive, negative and zero values of the quotient (Casey and Naghdi 1981a, 1983b, 1984a,b; Naghdi 1984a)

$$\frac{\hat{f}}{\hat{g}} = \Phi \quad (\text{say}), \tag{4.19}$$

so that

$$\begin{aligned} &(a) \quad \Phi > 0 \quad (\text{hardening}),\\ &(b) \quad \Phi < 0 \quad (\text{softening}),\\ &(c) \quad \Phi = 0 \quad (\text{perfectly plastic}). \end{aligned} \tag{4.20}$$

The relationships between the loading criteria (4.18) of the strain-space formulation and the associated conditions in stress space are summarized in Table 1. It is seen that during hardening behavior, the loading conditions in stress space and those in strain space imply one another. However, during softening and perfectly plastic behavior, no such equivalence exists; in this connection, see Fig. 4 of the present paper (which corresponds to Fig. 2 of Casey and Naghdi 1981a) illustrating the three types of material behavior defined by (4.20). It then follows that the stress-space and the strain-space formulations of plasticity are not equivalent.

For reasons that will be evident below, we observe that the expression for $\hat{g}$ defined by $(4.17)_1$ may be partially recast in terms of the normal $\partial f/\partial \boldsymbol{S}$ to the yield surface (4.11) in stress space rather than the normal $\partial g/\partial \boldsymbol{E}$ to (4.14). Thus, using $(4.17)_1$ and the relation between $\partial g/\partial \boldsymbol{E}$ and $\partial f/\partial \boldsymbol{S}$, we have[23]

$$\hat{g} = \left(\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}}\right)^T \left[\frac{\partial f}{\partial \boldsymbol{S}}\right] \cdot \dot{\boldsymbol{E}} = \frac{\partial f}{\partial \boldsymbol{S}} \cdot \left\{\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}}[\dot{\boldsymbol{E}}]\right\}. \tag{4.21}$$

Table 1
Relations between loading criteria in strain-space and associated conditions in stress space (with $g = f = 0$).

Strain-space loading criterion	Hardening ($\Phi > 0$)	Softening ($\Phi < 0$)	Perfectly plastic ($\Phi = 0$)
Unloading	$\hat{g} < 0 \Leftrightarrow \hat{f} < 0$	$\hat{g} < 0 \Rightarrow \hat{f} < 0$	$\hat{g} < 0 \Leftrightarrow \hat{f} < 0$
Neutral loading	$\hat{g} = 0 \Leftrightarrow \hat{f} = 0$	$\hat{g} = 0 \Leftrightarrow \hat{f} = 0$	$\hat{g} = 0 \Rightarrow \hat{f} = 0$
Loading	$\hat{g} > 0 \Leftrightarrow \hat{f} > 0$	$\hat{g} > 0 \Rightarrow \hat{f} < 0$	$\hat{g} > 0 \Rightarrow \hat{f} = 0$

[23] For details, see Casey and Naghdi (1984c).

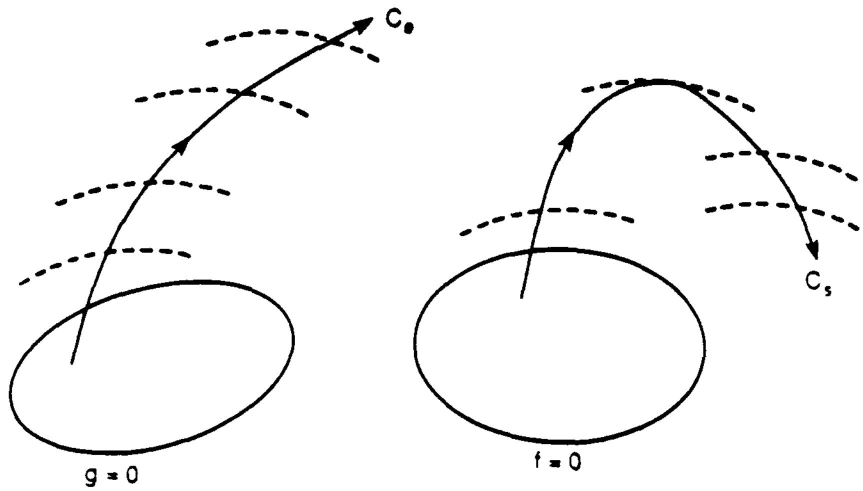


Figure 4
A sketch illustrating the motion of yield surfaces in strain space and stress space, which can be regarded as corresponding to an idealized stress-strain ($s - e$) diagram of a typical ductile metal in a uniaxial test. During loading the yield surface $\partial\mathscr{E}$ in strain space moves outwards with the strain trajectory C_e through positions indicated by the intersection of dashed curves and C_e. The corresponding yield surface $\partial\mathscr{S}$ in stress space moves outwards through positions indicated by the intersection of dashed curves and C_s during hardening behavior, is stationary during perfectly plastic behavior, and moves inwards during softening behavior.

If the quantity between the brackets { } in $(4.19)_2$ is denoted by $\boldsymbol{S}^*$ (say), then the scalar $\hat{g}$ when evaluated on $g = f = 0$ admits an interpretation (which is only an analogy) in stress space, namely that it represents the inner product of the normal to the yield surface in stress space and the tensor $\boldsymbol{S}^*$ which has the physical dimension of stress rate. It is easily seen from (4.13) and (4.21) that the loading criteria of strain space can be rewritten in terms of f, $\partial f/\partial \boldsymbol{S}$ and $\boldsymbol{S}^* = \{(\partial \boldsymbol{S}/\partial \boldsymbol{E})[\dot{\boldsymbol{E}}]\}$ which involves the rate of strain (not the rate of stress); and, despite the geometrical interpretation mentioned above, it should not be confused with criteria in stress space. Indeed, as noted previously (Casey and Naghdi 1984b,c), parts of plasticity theory can be stated interchangeably in terms of either stress or strain. For example, it is clear from (4.13) and the foregoing discussion that the yield surface in stress space and its normal can be transformed into the yield surface in strain space and its normal. Similarly, during elastic behavior, unloading and neutral loading (with $\dot{\boldsymbol{E}}_p = \dot{\kappa} = 0$) the tensor $\boldsymbol{S}^*$ coincides with $\dot{\boldsymbol{S}} = (\partial \boldsymbol{S}/\partial \boldsymbol{E})[\dot{\boldsymbol{E}}]$. During loading, however, $\boldsymbol{S}^*$ is only a part of $\dot{\boldsymbol{S}}$ and *cannot* always be obtained from $\dot{\boldsymbol{S}}$ (see Eqs. (14) and (44) of Casey and Naghdi 1984b). It may be emphasized that conditions of the type $(4.21)_2$ are characteristic of a strain-space formulation and should not, therefore, be regarded as stress-space loading criteria.

The nonequivalence of the strain-space and the stress-space formulations in plasticity is not an idle issue, inasmuch as the question of the significance of the strain-space formulation and the primacy of its loading criteria has been a source of misconception and has been viewed by some with skepticism. Some workers in the field accept certain features of the strain-space formulation (alas interpreting it as belonging to the stress-space formulation), but do not appear to recognize the full significance of the strain space setting and its implications discussed in this subsection. For example, with the use of special constitutive equations, loading criteria involving S^* have been used in the computational literature (Hughes 1983). In fact, Hughes (1983) dismisses the strain-space formulation as unnecessary, yet the loading criteria and constitutive equations used by him presuppose a knowledge of the rate of strain rather than the rate of stress[24]. Similarly, Yoder and Iwan (1981) have claimed that the stress-space and the strain-space formulations of plasticity are equivalent. This was refuted in a discussion of their (Yoder and Iwan 1981) paper by Casey and Naghdi (1982) and need not be repeated here; for a detailed analysis on the issue of "nonequivalence," which also bears on the misleading statements of Yoder and Iwan, see Casey and Naghdi (1983c) and Naghdi (1984a).

Another example of misconception regarding the significance of the strain-space formulation and its predictive capabilities has surfaced in connection with the curious softening responses that have been observed in uniaxial compression tests of geological materials (Wawersik and Fairhurst 1970). To elaborate, we recall that the phenomenon known as "critical softening" (or "perfectly brittle" behavior) is typified by a vertical stress-strain curve in a uniaxial compression; and "subcritical" and "normal" softening correspond, respectively, to stress-strain curves which fall to the left and to the right of the vertical. Because of the verticality of the stress-strain response in a uniaxial compressive stress test, it may at first appear that critical softening corresponds to $\dot{E}$ being zero[25] and consequently to the vanishing of $\hat{g}$ defined by $(4.17)_1$. Such a viewpoint is inherent in the argument presented by Dafalias (1984a, p. 157), and would imply that the critical softening during loading cannot be dealt with in the strain-space formulation of plasticity. However, Casey and Lin (1986) have shown that critical, subcritical and normal softening can be accommodated within the scope of the strain-space formulation with $\hat{g} > 0$ while loading is taking place. The relevant definition in the development of Casey and Lin (1986) hinges on the sign or vanishing of the quantity $\dot{S} \cdot \dot{E}$ (rather than just $\dot{E}$) during loading.

[24] For additional remarks on this point see Casey and Naghdi (1984c, pp. 60–62).
[25] It should be observed that even in the case of uniaxial stress although the strain component $E_{11} = 0$, the remaining strain components in the lateral directions of the specimen are not necessarily zero.

4G. *Flow rule*

The term flow rule in most of the literature of plasticity refers to a constitutive equation for plastic strain rate. In a general theory of plasticity, it is usual to assume that the rate of plastic strain can be expressed as a linear function of the rate of strain (or the rate of stress) with coefficient which is independent of rate quantities but may depend on variables such as $\mathscr{V}$ (or $\mathscr{Y}$). Thus, for example, in a strain-space setting and in the context of the Langrangian formulation of the theory the rate of plastic strain has the form

$$\dot{\boldsymbol{E}}_p = \begin{cases} \text{function of variables } \mathscr{V})\dot{\boldsymbol{E}}, & \text{during loading defined by (4.16d)},\\ \boldsymbol{0}, & \text{otherwise}^{26}. \end{cases} \tag{4.22}$$

Alternatively, in a stress-space setting but again in the Lagrangian form of the theory the right-hand side of an equation similar to (4.22) would be linear in $\dot{\boldsymbol{S}}$ and with coefficient which is a function of the variables $\mathscr{Y}$.

Recent and current literature representing efforts of the majority of the various schools of plasticity are directed toward an Eulerian formulation of the theory constructed in a stress-space setting. The preference for the Eulerian formulation is evidently based on one or both of the following presuppositions: (i) the belief, founded perhaps in analogy with viscous fluid flow, that such formulations are more relevant to large elastic-plastic deformations; and (ii) the view that the construction of the theory in terms of the Cauchy stress and its rate is more fundamental. Most workers who share the preference for the Eulerian version of the theory, begin by considering the decomposition of the velocity gradient $\boldsymbol{L}$ or rate of the deformation tensor $\boldsymbol{D}$ into additive "elastic" and "plastic" parts $\boldsymbol{L}_e$ and $\boldsymbol{L}_p$ or $\boldsymbol{D}_e$ and $\boldsymbol{D}_p$, respectively, so that[27]

$$\boldsymbol{L} = \boldsymbol{L}_e + \boldsymbol{L}_p, \qquad \boldsymbol{D} = \boldsymbol{D}_e + \boldsymbol{D}_p. \tag{4.23}$$

Subsequently, they prescribe a constitutive equation for $\boldsymbol{D}_p$ in terms of a rate of Cauchy stress; the rate operator here is not the usual material derivative but an objective rate such as the corotational (or Jaumann) rate[28] which renders the stress rate properly invariant under s.r.b.m. Thus, the constitutive equation for $\boldsymbol{D}_p$ will have the form

$$\boldsymbol{D}_p \propto \text{an objective rate of } \boldsymbol{T}, \tag{4.24}$$

[26] By the word "otherwise," we mean the conditions (4.18a,b,c) for an elastic state, unloading and neutral loading.

[27] Some authors writing in the context of a macroscopic theory also decompose the vorticity (or the spin) tensor into elastic and plastic parts (see, for example, Nemat-Nasser 1983, Dafalias 1984b, Dienes 1986, and Agah-Tehrani *et al.* 1987).

[28] For a discussion of objective rates in continuum mechanics, see Truesdell and Toupin (1960, Secs. 148–150) and Truesdell and Noll (1965, Sec. 16).

with coefficient function which may depend on $\boldsymbol{T}$, as well as $\boldsymbol{E}$, and $\boldsymbol{E}_p$ (or $\boldsymbol{F}$ and $\boldsymbol{F}_p$). It should be noted at this point that various schools of plasticity which embrace the Eulerian approach just described do not all agree on the same objective rate; and this, in turn, has given impetus in recent years to a variety of proposals for the choice of rate of stress and the rate of other variables in the Eulerian formulation (see, e.g., Atluri 1984; Dafalias 1983; Dienes 1979, 1986; Lee *et al.* 1983; Loret 1983; Nemat-Nasser 1983, among others).

Some justifiable criticism of the foregoing Eulerian approach and a constitutive equation of the type (4.24) is as follows:

(1) Remembering the relation between $\boldsymbol{L}$ and the rate of deformation gradient, as well as the decomposition (4.1), it follows that not all three quantities $\boldsymbol{L}$, $\boldsymbol{L}_e$, $\boldsymbol{L}_p$ can be simply related (in the form $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$) to the rate of their respective parts in (4.1). For example, if $\boldsymbol{L}_e$ is taken in the form $\boldsymbol{L}_e = \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1}$, then necessarily $\boldsymbol{L}_p$ will not have the same form; and, similarly, if $\boldsymbol{L}_p$ is specified in the form $\boldsymbol{L}_p = \dot{\boldsymbol{F}}_p\boldsymbol{F}_p^{-1}$, then $\boldsymbol{L}_e$ will not have the desired form. This suggests an essential arbitrariness in the decomposition $\boldsymbol{L} = \boldsymbol{L}_e + \boldsymbol{L}_p$ and the extent to which the results of a theory are influenced by this arbitrariness remains unclear. A parallel remark applies to $\boldsymbol{D}_e$ and $\boldsymbol{D}_p$ in the decomposition $(4.23)_2$.

(2) Presumably, $\boldsymbol{D}_p$ is calculated from $\boldsymbol{L}_p$ and is given by a relation of the same form as $(2.3)_2$, while $\boldsymbol{D}_p$ is related to $\dot{\boldsymbol{E}}_p$ by a relation of the form $(2.3)_1$ with $\boldsymbol{F}$ replaced by $\boldsymbol{F}_p$. But, is this an acceptable physics for characterization of the flow rule?

(3) At best it seems that the decompositions $(4.23)_{1,2}$ are a generalization of corresponding expressions in infinitesimal plasticity, since (to the order of approximation) in the linear theory the expression $(4.23)_2$ would be identical to the rate of strain = rate of (elastic part + plastic part). A parallel comment holds in regard to $(4.23)_1$.

The Eulerian formulation of the flow rule discussed in the preceding two paragraphs is based on $(4.23)_{1,2}$. In a related development to some of his earlier papers (e.g., Lee 1969 and Lubarda and Lee 1981), Lee (1981, Sec. 2) adopts a different procedure. He begins by recalling the decomposition (4.1), defines the strain measures $(4.2)_{1,2}$, identifies the factor $\boldsymbol{F}_e$ with the right polar decomposition $\boldsymbol{F}_e = \boldsymbol{R}_e\boldsymbol{V}_e$; and then, with the rotation $\boldsymbol{R}_e = \boldsymbol{I}$, he specifies $\boldsymbol{F}_e$ = the stretch $\boldsymbol{V}_e$ and arrives at the expression $\boldsymbol{L}_e = \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1}$. Even with this approximation, Lee himself observes that $\boldsymbol{D} \neq \boldsymbol{D}_e + \boldsymbol{D}_p$; see Eq. (2.17) of Lee (1981). However, next he states that "... the elastic strain in metals is usually small, $\sim 10^{-3}$, $\boldsymbol{V}_e = \boldsymbol{I} + \boldsymbol{\delta}$ where $\boldsymbol{\delta} \sim 10^{-3}$, so that by neglecting $\boldsymbol{\delta}$ ..." he approximates $\boldsymbol{D}$ and obtains an expression in the form $(4.23)_2$. Fundamentally, there is hardly any difference between Lee's (1981) line of argument and an *ad hoc* approach which motivates $(4.23)_2$ as a

generalization of the usual expression for the total strain rate in infinitesimal plasticity (see item (3) in the preceding paragraph). Moreover, such approximate schemes as that advocated by Lee (1981) avoids the issue—even aside from violation of invariance under s.r.b.m. associated with his choice $\boldsymbol{F}_e = \boldsymbol{V}_e$ (see subsection 4C). Even if this type of approximation is proven successful in some special cases it will not ensure its appropriateness in the construction of a general theory of finite plasticity, which after all must be valid for all types of motion and not just for certain special motions. Also, it should be noted that despite his statement regarding his approximations for metals (Lee 1981), in a later paper Lee (1984, p. 234) himself recognizes some exceptions.

Because of the important status that is relegated to the structure of the constitutive equations representing the flow and hardening rules within the framework of a general finite theory of elastic-plastic materials, it is desirable to comment further on the nature of some related developments pertaining to the constitutive equations for the flow rule (e.g., Nemat-Nasser 1983, Agah-Tehrani *et al.* 1987, among others). The developments in these papers, while not identical to the Eulerian format of the references cited earlier in this subsection, advocate points of view that are akin to the decomposition of the type $(4.23)_{1,2}$ and a flow rule of the form (4.24). Thus, in a discussion which employs macroscopic variables, Nemat-Nasser (1983, Sec. 3) begins with the decomposition of the rate of the deformation $\boldsymbol{D}$ and the spin $\boldsymbol{W}$ [see his Eqs. (3.2) to (3.4)] in the form[29]

$$\boldsymbol{D} = \boldsymbol{D}_* + \boldsymbol{D}_p, \qquad \boldsymbol{W} = \boldsymbol{W}_* + \boldsymbol{W}_p, \tag{4.25}$$

identifies the quantities $\boldsymbol{D}_*$ and $\boldsymbol{W}_*$ as "the accommodating elastic contribution," and specifies a constitutive equation for a "stress change" by an equation which relates the Jaumann rate of the Kirchhoff stress ($= \varrho \boldsymbol{T}/\varrho_0$) linearly to $\boldsymbol{D}_*$ with his Jaumann rate (see his Eq. $(3.4)_2$) defined as Jaumann rate of $(\ ) = [(\ ) - \boldsymbol{W}_*(\ ) + (\ )\boldsymbol{W}_*]$. He then obtains a constitutive equation for the rate of plastic deformation $\boldsymbol{D}_p$ which is more special than the form (4.24). The development in Nemat-Nasser's (1983) paper, along with his constitutive equation for the rate of hardening (to be considered in the next subsection 4H), does not appear to be of any help towards clarifying the relevant issues. In fact, the relation of his development (Nemat-Nasser 1983) to the work of others is ambiguous. Similarly, a recent paper by Agah-Tehrani, Lee, Mallet and Onat (1987) represents still another attempt towards the clarification of the unresolved issues in finite plasticity. Employing a combination of a variety of kinematical variables already discussed in subsections 4A, 4C and 4G, Agah-Tehrani *et al.* admit

[29] The notations $\boldsymbol{D}_*$ and $\boldsymbol{W}_*$ in (4.25) correspond to the symbols $\boldsymbol{D}^*$ and $\boldsymbol{W}^*$ in Nemat-Nasser's (1983) paper.

(see their Eq. (1)) a polar decomposition of the deformation gradient that implicitly incorporates an approximative scheme which violates invariance under s.r.b.m. associated with their choice of $\boldsymbol{F}_e = \boldsymbol{V}_e$ (a choice that has been already criticized in subsection 4C). This is followed by the introduction of a kinematical variable identified as the "plastic" strain rate $\hat{\boldsymbol{D}}_p$ which in their words (Agah-Tehrani *et al.* 1987, Eq. (2) and the statements on p. 320) "... expresses the component of strain rate in the current elastically-plastically deformed configuration associated with $\boldsymbol{D}_p$ in the intermediate unstressed configuration based on the plastic deformation $\boldsymbol{F}_p$." Their flow rule is subsequently stated in terms of $\hat{\boldsymbol{D}}_p$. It does not appear that this development for the flow rule represents any improvement over and above those discussed in the preceding paragraph of this subsection. We postpone comments on the development of the hardening rule in the paper of (Agah-Tehrani *et al.* 1987) until the next subsection 4H. Again, with reference to the constitutive equation for a flow rule, mention should be made of the fact that variants of the decompositions (4.23) along with a flow rule of the form (4.24) are being opted in a large number of recent papers (e.g., Lee 1987, Anand and Lush 1987) and these are subject to the same criticisms discussed in the preceding two paragraphs.

4H. *Hardening rule*

The term hardening rule refers to a constitutive equation for the rate of the work-hardening parameter κ. In the context of the Lagrangian formulation of a general theory of the type considered in the papers of Green and Naghdi (1965, 1966) and Naghdi and Trapp (1975a), the rate of κ is assumed to be linear in $\dot{\boldsymbol{E}}_p$ with coefficient function which depends on the variables $(4.7)_1$ in the strain-space formulation[30]. But, in view of the assumed form (4.22), the rate of κ can be expressed as a linear function of $\dot{\boldsymbol{E}}$ in the form[31]

$$\dot{\kappa} = \begin{cases} (\text{function of variables } \mathscr{V})\dot{\boldsymbol{E}}, & \text{during loading defined by (4.16d)}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.26}$$

A general form for the loading function in both the stress and strain spaces has been already indicated in this section (see subsection 4E); however, less general forms may be assumed for a special class of materials, as

[30] The coefficient function in the stress-space formulation depends on the variables (4.10) but the result can be expressed in terms of a different function of $(4.7)_1$ with the use of (4.6).

[31] Constitutive equations of the form (4.26) include as a special case the classical characterization of work-hardening in terms of plastic work or equivalent plastic strain (Hill 1950, pp. 26–30). The word "otherwise" in $(4.26)_2$ has the same meaning as that in $(4.22)_2$.

in the Prandtl-Reuss type constitutive equations for elastic-perfectly plastic solids. The assumed special form of the loading function depends largely on one's objective either in describing certain physical phenomenon (e.g., the Bauschinger effect, or the material response to cyclic loading, etc.) or in the context of particular application. To elaborate on this point, it is best to consider a special choice for the form of the loading function and then illustrate in the presence of small deformation certain features of both the loading functions g and f, as well as those of the work-hardening scalar κ. To this end, let (γ, γ_p, τ) be the deviatoric parts of $(\boldsymbol{E}, \boldsymbol{E}_p, \boldsymbol{S})$, respectively, denote their spherical parts by $(\bar{e}\boldsymbol{I}, \bar{e}_p\boldsymbol{I}, \bar{s}\boldsymbol{I})$, and temporarily specify a loading function in the context of infinitesimal plasticity in the form

$$\begin{aligned} f &= \left[\tau - \frac{\bar{\alpha}(\kappa)}{2}\gamma_p\right]\cdot\left[\tau - \frac{\bar{\alpha}(\kappa)}{2}\gamma_p\right] - \kappa \\ &= 4\mu^2\left[\gamma - \left(1 + \frac{\bar{\alpha}(\kappa)}{4\mu}\right)\gamma_p\right]\cdot\left[\gamma - \left(1 + \frac{\bar{\alpha}(\kappa)}{4\mu}\right)\gamma_p\right] - \kappa = g, \end{aligned} \tag{4.27}$$

where the expression $(4.27)_2$ for g is obtained from f with the use of generalized Hooke's law, μ is the shear modulus of elasticity and $\bar{\alpha}$ may depend on[32] κ. The loading function f specified by $(4.27)_1$ is a generalization of the von Mises yield function in stress space and tacitly incorporates the assumption that it is independent of the mean normal stress $\bar{s}$. With $\bar{\alpha} = 0$ and $\kappa =$ constant, $(4.27)_1$ becomes identical to the von Mises yield function for an elastic-perfectly plastic solid[33]. It is of interest to note that for the special case in which $\bar{\alpha} = 0$ and $\kappa =$ constant the yield condition (4.11) with f given by $(4.27)_1$ is stationary, while the corresponding yield condition (4.14) with g given by $(4.27)_2$ would still depend explicitly on γ_p.

Returning to (4.27) with neither κ nor $\bar{\alpha}$ necessarily a constant, the following rough geometrical observations may be made: (i) the size of the loading surface at any instant (which represents the extent of the elastic region) is determined by the work-hardening parameter κ, (ii) the factor $1/2\bar{\alpha}\gamma_p$ may be interpreted as the center of the yield surface in the deviatoric stress space, (iii) the quantity $\{1 + (1/4\bar{\mu})\bar{\alpha}\}\gamma_p$ may be interpreted (in analogy with a sphere in 3-space) as the center of the yield surface in deviatoric strain space, and (iv) the translation of the yield surface in stress space is represented by $(\bar{\alpha}/2)\gamma_p$. Clearly, if $\bar{\alpha}$ is specified to be nonzero but only a constant, then at a given value of the plastic strain (here γ_p) the translation of the yield surface is determined independently of the loading history. On

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32 In writing (4.27), we have used the symbol $\bar{\alpha}$ in order to reserve the corresponding symbol without an overbar for later use. As will be brought out presently, a more general approach is to introduce a second order tensor $\boldsymbol{\alpha}$ (in place of $(\bar{\alpha}/2)\gamma_p$) as an independent variable with its own rate-type constitutive equation.

33 The constant κ here corresponds to k^2 in the usual statement of Mises yield condition, k being the yield limit in simple shear.

the other hand, one may account for the effect of loading history on the translation of the yield surface by regarding $\bar{\alpha}$ to depend on κ as displayed in (4.27)[34].

The discussions in the last two paragraphs easily suggest a less restrictive loading function simply by replacing the quantity $(\bar{\alpha}/2)\gamma_p$ in each of the square brackets in $(4.26)_1$ by a shift tensor α which depends linearly on plastic strain with a coefficient which may depend on[35] κ. But this is still too special. Returning to finite plasticity, as was noted earlier a more general approach would be to regard the shift tensor (or the 'back stress') as an independent variable defined by a constitutive equation (of the Prager-Ziegler type[36]) for the evolution of α in the Lagrangian formulation of finite plasticity by

$$\dot{\alpha} = (\text{function of variables } \mathscr{V} \text{ and } \alpha)\dot{E}. \tag{4.28}$$

By contrast, in the Eulerian formulation of the theory, a constitutive equation corresponding to (4.27) may be stated as[37]:

$$\text{an objective rate of } \alpha = (\text{function of } \boldsymbol{T}, \boldsymbol{F}, \boldsymbol{F}_p, \kappa, \alpha)\boldsymbol{D}_p. \tag{4.29}$$

In connection with the objective rate on the left-hand side of (4.29), a number of authors who prefer the Eulerian formulation have assigned a preferred status to one particular rate or another. With reference to the spurious oscillatory shearing stress response to a monotonic simple shearing observed by Nagtegaal and DeJong (1982), who specified their hardening rule by an expression which can be categorized as a very special case of the form (4.29) with the operator on the left-hand side being the Jaumann derivative of $\alpha = \dot{\alpha} - \boldsymbol{W}\alpha + \alpha\boldsymbol{W}$, several authors have proposed to replace the corotational (or Jaumann) rate in such constitutive equations as (4.29) by some other special objective rate (see, e.g., Dienes 1979 and 1986, Lee *et al.* 1983, Dafalias 1983, Nemat-Nasser 1983, Onat 1984, among others). For example, Nemat-Nasser (1983, Sec. 3) in his discussion of hardening rule attempts to partly justify his choice of a particular objective rate on the left-hand side of his expression for the rate of the back stress (see his Eqs. (3.20) and (3.24)) on the basis of some special consideration of micromechanical effects. Equally troublesome is a different line of argument put

[34] In fact, only a linear dependence of $\bar{\alpha}$ on κ will suffice for describing (with good accuracy) material response during stress and strain cycling in a uniaxial homogeneous deformation (Naghdi and Nikkel 1984).

[35] A constitutive equation for the shift tensor α which depends linearly on plastic strain was evidently first proposed by Kadashevich and Novozhilov (1958, Eq. (1.10)). A similar assumption has been utilized by Caulk and Naghdi (1978), Naghdi and Nikkel (1984, 1986) and Dogui and Sidoroff (1985).

[36] See Prager (1955, 1956) and Ziegler (1959) who considered a constitutive equation for α only in the context of rigid-plastic materials.

[37] Actually the various developments in the Eulerian formulation employ special variants of (4.29). For example, Lee *et al.* (1983, Eq. (7)) for the coefficient on the right-hand side of (4.29) assume a function of the equivalent plastic strain.

forward in a recent paper of Agah-Tehrani, Lee, Mallet and Onat (1987): First, they recall the spurious observation of Nagtegaal and DeJong (1982) and by transferring the last two terms of the Jaumann derivative to the right-hand side, they record the hardening rule of Nagtegaal and DeJong in the form

$$\dot{\boldsymbol{\alpha}} = (\text{function of an equivalent plastic strain})\, \boldsymbol{D}_p + \boldsymbol{W}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{W}.$$

Next, they observe the anomalous behavior of the above expression and this suggests to them the remedy that the hardening rule should take the form (see their Eqs. (5) and (6))[38]

$$\dot{\boldsymbol{\alpha}} = (\text{function of an equivalent plastic strain})\, \boldsymbol{D}_p + \boldsymbol{W}_{*}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{W}_{*}, \tag{4.30}$$

where $\boldsymbol{W}_{*}$ "is the deformation imposed angular velocity of the embedded back stress . . . ." This vague definition is later supplemented (see their Eqs. (31), (32) and (34)) by the specification of $\boldsymbol{W}_{*} = \boldsymbol{W} + \hat{\boldsymbol{W}}$, where the skew-symmetric $\hat{\boldsymbol{W}}$ is a tensor function of $\boldsymbol{\alpha}$ and the "plastic" strain rate $\hat{\boldsymbol{D}}_p$ introduced in the last paragraph of the last subsection. In effect, the hardening rule (4.30) is of the same form as (4.29) with part of the special objective rate transferred to the right. The stipulation that any particular rate in the development of a constitutive theory should enjoy such a preferred status is questionable and requires further consideration as will be discussed in the next subsection.

4I. *Lagrangian versus Eulerian descriptions*

It is well-known that the theory of nonlinear elastic materials can be formulated either in terms of the Cauchy stress tensor $\boldsymbol{T}$ or the symmetric Piola-Kirchhoff stress tensor[39] $\boldsymbol{S}$. In fact, in the context of nonlinear elasticity any question of preference for either Lagrangian or Eulerian formulation can be immediately disposed of since the two formulations can be brought into correspondence through the transformation (3.1). By contrast, the preference for the use of the Cauchy stress tensor $\boldsymbol{T}$ (rather than the symmetric Piola-Kirchhoff stress $\boldsymbol{S}$) in finite plasticity has surfaced in the literature from time to time (see e.g., the discussion by Onat of the paper of Green and Naghdi 1966, p. 131; and the papers of Lee 1969 and Palgen and Drucker 1983, among others).

The preference of several schools of plasticity for adopting the Eulerian formulation, including a special status for a particular objective rate in the

[38] The use of the temporary notation $\boldsymbol{W}_{*}$ in (4.30) with reference to the paper of Agah-Tehrani *et al.* (1987) should not be confused with the same symbol introduced to represent a different quantity (with reference to a paper of Nemat-Nasser 1983) in the last paragraph of the previous subsection 4G.

[39] By an elastic material we mean one for which a potential is assumed to exist (Green, G. 1839). Such a material in the current literature is sometimes referred to as Green-elastic or hyper-elastic. In this connection, see Truesdell and Noll (1965, pp. 13 and 119).

expressions of the rate of Cauchy stress in the flow rule (4.24) and the rate of the shift tensor in (4.29), was indicated previously in this section (see subsections 4D, 4G and 4H). Motivated by a desire to clarify this issue, Casey and Naghdi (1988) have examined the relationship between the Lagrangian and Eulerian strain-space formulations of finite plasticity with the limitation to rigid-plastic materials. This restriction to rigid-plastic materials is made for conceptual simplicity, although it is accompanied by a general ("anisotropic") hardening law. It is then demonstrated that the Lagrangian and Eulerian formulations are equivalent and that the choice of an objective rate is immaterial. This conclusion conflicts with the procedure adopted in a number of recent papers pertaining to Eulerian formulations in which preference is given to one particular objective rate or another. Thus, it becomes clear that the difficulties which have been encountered in the past are due not to the special choice of the rate of stress or rate of the shift tensor, but rather to the restrictive nature of the constitutive assumptions[40]. Indeed once sufficiently general constitutive relations are assumed, *all* objective rates are acceptable.

The outcome of Eulerian versus Langrangian descriptions for rigid-plastic materials (Casey and Naghdi 1988) is not fully conclusive until it is substantiated in a more general setting; it is being currently studied (by Casey and Naghdi) in the context of elastic-plastic materials and the possibility of extending the nature of the previous (Casey and Naghdi 1988) conclusions is being investigated.

5. A strain-based formulation of plasticity

The detailed review of "the state of the art" in section 4 spells out the nature of disagreements and the implication of the varied starting assumptions in the construction of a general theory of finite plasticity based entirely on a stress-based formulation. It is very likely that it may be some time before the differences in basic viewpoints regarding the issues addressed in section 4 will be fully resolved.

None of the existing developments in finite elastic-plastic materials have so far been able to provide a complete definition of plastic strain to any degree of finality, even though one or more developments may be preferred depending on one's viewpoint and taste. Apart from this, it is evident from the detailed discussions in the subsections 4A to 4I of the previous section that the strain-space formulations of the theory are free from various special

[40] To see this, consider for example a restrictive form of (4.29) with its left-hand side specified by the Jaumann rate of $\mathbf{a}$ and with the coefficient function on its right-hand side being just a constant. Then, it can be easily verified that such a special form of (4.29) is not form-invariant under change to another objective rate.

anomalous behavior that has accompanied the developments of the Eulerian version of plasticity based wholly on a stress-space formulation. Moreover, a close examination of the strain-space formulation exhibits agreeable features that share commonality with other physically acceptable macroscopic theories of material behavior. Given this background and the fact that enough justification for a strain-based formulation is indicated in section 4, it seems best to discuss the main results of the strain-space formulation and the final steps of its development (as is currently understood) in a separate section. Thus, we include here a summary of the basic constitutive results of the strain-space formulation of a purely mechanical theory of elastic-plastic materials, a discussion of some of its important features and special cases, as well as a rapid account of developments pertaining to constitutive restrictions that have been derived from a work inequality over a strain cycle.

5A. *A system of Lagrangian constitutive equations for finite plasticity*

In view of the remarks in section 4, it is clear that the insistence (in some of the literature of the past two decades) that a meaningful finite theory must necessarily be formulated only in Eulerian form is unfounded on physical grounds. Keeping this background in mind and given the primacy of the strain-space formulation (proposed in subsection 4F), we now choose to further elaborate on an existing Lagrangian formulation of finite plasticity which possesses considerable conceptual simplicity. Thus, we summarize below the basic constitutive ingredients of a strain-space formulation of the purely mechanical theory developed by the present writer and co-workers during the past two decades of the original thermodynamical theory of Green and Naghdi (1965, 1966).

We begin by recalling a constitutive assumption for the stress response in the form (4.12) and its inverted form (4.9), but now also enlarge the number of independent variables in order to include the additional second order symmetric tensor variable $\boldsymbol{\alpha}$ introduced in subsection 4H (the paragraph containing (4.28)). Thus, we write

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\mathscr{U}) = \boldsymbol{S}(\bar{\mathscr{U}}), \qquad \boldsymbol{E} = \hat{\boldsymbol{E}}(\mathscr{V}), \tag{5.1}$$

where the abbreviations $\mathscr{U}$, $\bar{\mathscr{U}}$ and $\mathscr{V}$ now stand for

$$\mathscr{U} = (\boldsymbol{E}, \mathscr{W}), \qquad \bar{\mathscr{U}}(\boldsymbol{E} - \boldsymbol{E}_p, \mathscr{W}), \tag{5.2}$$

$$\mathscr{V} = (\boldsymbol{S}, \mathscr{W}) \tag{5.3}$$

and

$$\mathscr{W} = (\boldsymbol{E}_p, \kappa, \boldsymbol{\alpha}). \tag{5.4}$$

It should be noted that the shift tensor $\boldsymbol{\alpha}$ (or the 'back stress') has the same invariance property under s.r.b.m. as the plastic strain $\boldsymbol{E}_p$, i.e., $\boldsymbol{\alpha}^+ = \boldsymbol{\alpha}$. Also, the yield functions g and f are now, respectively, functions of the variables $(5.2)_1$ and (5.3). Similarly, the function Φ defined by (4.19) and other relevant quantities now depend on the variables $(5.2)_1$. It is clear that if the effect of $\boldsymbol{\alpha}$ is suppressed, then the variables $(5.2)_{1,2}$ and (5.3) reduce to those defined by $(4.7)_{1,2}$ and (4.10), respectively, and similarly the response functions for $\boldsymbol{S}$ and $\boldsymbol{E}$ reduce to (4.12) and (4.9). Next, from the assumptions of the forms (4.22), (4.24) and (4.28) for $\dot{\boldsymbol{E}}_p$, $\dot{\kappa}$, and $\dot{\boldsymbol{\alpha}}$, as well as the "consistency" condition that during loading the strain trajectory C_e (introduced in subsection 4F) remains on the yield surface $\partial\mathscr{S}$ so that $\dot{g} = 0$ (with $g = 0, \hat{g} > 0$), follow the reduced expressions[41]

$$\dot{\boldsymbol{E}}_p = \boldsymbol{\varrho}\hat{g}, \qquad \dot{\kappa} = \lambda\hat{g}, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{\beta}\hat{g} \qquad (g = 0, \hat{g} \geq 0) \tag{5.5}$$

and the "consistency" condition

$$1 + \lambda\frac{\partial g}{\partial \kappa} + \frac{\partial g}{\partial \boldsymbol{E}_p}\cdot\boldsymbol{\varrho} + \frac{\partial g}{\partial \boldsymbol{\alpha}}\cdot\boldsymbol{\beta} = 0, \tag{5.6}$$

where $\boldsymbol{\varrho}$ and $\boldsymbol{\beta}$ are symmetric second order tensor-valued functions of $\mathscr{V}$ and λ is a scalar-valued function of $\mathscr{V}$. Equations $(5.5)_{1,2,3}$, as well as $\dot{\boldsymbol{E}}_p = \boldsymbol{0}$, $\dot{\kappa} = 0$, $\dot{\boldsymbol{\alpha}} = \boldsymbol{0}$ for states specified by the loading criteria (4.18a,b,c), constitute the *flow and the hardening rules* of the strain-space formulation of finite plasticity. Then, in addition to the response function $\hat{\boldsymbol{S}}$ in (5.1), an elastic-plastic material is completely specified by four functions g, λ, $\boldsymbol{\varrho}$ and $\boldsymbol{\beta}$ subject to the restriction (5.6). In fact, if $\partial g/\partial\kappa \neq 0$, (5.6) may be used to solve for λ.

Returning to the basic constitutive developments (in the early part of this subsection), it should be emphasized that the constitutive ingredients represented by the stress response $(5.1)_1$ and the four functions g, λ, $\boldsymbol{\varrho}$ and $\boldsymbol{\beta}$ are fully general for characterization of the rate-independent behavior of elastic-plastic materials. But for most or many applications a somewhat simpler form of these would be sufficient. It is, therefore, desirable to simplify these and provide physically acceptable restrictions to be placed on the various constitutive response functions, which are discussed in subsection 5C.

The constitutive results summarized in this subsection, in view of the presence of $\boldsymbol{\alpha}$ in (5.4), possess a more general structure than those considered in section 4 prior to subsection 4H. However, while the addition of a variable such as $\boldsymbol{\alpha}$ and its evolution equation $(5.5)_3$ are useful ingredients, they do not alter the basic structure of the theory. For this reason and in

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[41] Our notation here is patterned after that used recently by Casey and Naghdi (1984b,c), where the relationship between the symbols in (5.5)–(5.6) and corresponding results in earlier papers is indicated.

order to continue the discussion in as simple a manner as possible, in the rest of this section and most of the developments that follow in sections 6–8 the effect of $\boldsymbol{\alpha}$ in (5.4) and hence in various constitutive response functions will be suppressed.

5B. *Some special cases of the general theory*

In this subsection, we elaborate on some noteworthy features of the basic constitutive results (5.1)–(5.6) in the absence of the variable $\boldsymbol{\alpha}$ and also discuss important special cases of the general theory[42].

(1) The definition (4.19) holds during loading from an elastic-plastic state ($g = f = 0$). It then follows from (4.19) and the conditions (4.20) that during loading in a region of hardening or softening the reduced constitutive equations $(5.5)_{1,2}$ may be expressed in terms of $\hat{f}$ as

$$\dot{\boldsymbol{E}}_p = \frac{\hat{f}}{\Phi}\boldsymbol{\varrho}, \qquad \dot{\kappa} = \frac{\hat{f}}{\Phi}\lambda, \tag{5.7}$$

and these can be regarded as the induced stress-space flow and hardening rules for loading in a region of hardening or softening. In a region of perfectly plastic behavior, $\dot{\boldsymbol{E}}_p$ and $\dot{\kappa}$ cannot be expressed in terms of $\hat{f}$ and must be calculated directly from $(5.5)_{1,2}$.

(2) The invertibility issue of the main constitutive results, i.e., the invertibility of the stress rate $\dot{\boldsymbol{S}}$ and the strain rate $\dot{\boldsymbol{E}}$ represent an important aspect of the general theory. For example, in some boundary-value (or initial-value) problems, it may be desirable to calculate $\dot{\boldsymbol{S}}$ in terms of a specified history of deformation and hence $\dot{\boldsymbol{E}}$. For this purpose, a detailed examination of the expression for $\dot{\boldsymbol{S}}$ along any strain trajectory C_e during loading can be reduced to the form

$$\dot{\boldsymbol{S}} = \mathscr{K}\mathscr{L}[\dot{\boldsymbol{E}}] \tag{5.8}$$

and its inverse

$$\dot{\boldsymbol{E}} = \mathscr{L}^{-1}\mathscr{K}^{-1}[\dot{\boldsymbol{S}}], \tag{5.9}$$

where $\mathscr{L} = \partial\hat{\boldsymbol{S}}/\partial\boldsymbol{E}$ and

$$\mathscr{K} = \mathscr{I} + \hat{\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{S}} \tag{5.10}$$

is a dimensionless fourth order tensor which has a sixth order determinant and satisfies the relation

$$\det \boldsymbol{K} = \Phi \text{ on the right-hand side of (4.19).} \tag{5.11}$$

[42] We discuss here only a few of these. For details and additional results see Casey and Naghdi (1984b).

Also, in (5.10)–(5.11), the symbol $\otimes$ denotes tensor product, $\mathscr{I} = (1/2)(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM})\boldsymbol{e}_K \otimes \boldsymbol{e}_L \otimes \boldsymbol{e}_M \otimes \boldsymbol{e}_N$, δ_{KL} is the Kronecker delta representing the components of the fourth order unit tensor referred to basis $\boldsymbol{e}_K \otimes \boldsymbol{e}_L$ and the symmetric second order tensor $\hat{\boldsymbol{\sigma}}$ is defined by

$$\hat{\boldsymbol{\sigma}} = \lambda \frac{\partial \boldsymbol{S}}{\partial \kappa} + \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}_p}[\boldsymbol{\varrho}] = \hat{\boldsymbol{\sigma}}^T. \tag{5.12}$$

The tensor $\mathscr{K}$ contains contributions from all of the fundamental properties embodied in the rate-independent theory, namely the functions $\boldsymbol{\varrho}$, λ and g (or f). In this connection, it is particularly noteworthy that the quantity $\Phi(=\hat{f}/\hat{g})$ in (5.11)—which is directly measurable in an experiment—can vary as a function of deformation to which the material is subjected; and that, at a given time, assumes different values at different points in strain and stress spaces[43].

(3) The nature of a finite theory of rigid-plastic materials ($\boldsymbol{E} = \boldsymbol{E}_p$) developed by Casey (1986) as a limiting case of the strain-space formulation of plasticity summarized in the previous paragraphs of this subsection. By considering the limiting case as $\boldsymbol{E} - \boldsymbol{E}_p \to 0$ of the general theory of elastic-plastic materials with all dependent variables regarded as functions of the variables $\hat{\mathscr{U}}$ defined by $(4.7)_2$, the loading criteria can be reduced to

$$\begin{aligned} &\text{(a)}\ \ \dot{\boldsymbol{E}} = 0 \quad \text{and } \lim f \le 0 \quad \text{(nonloading)},\\ &\text{(b)}\ \ \dot{\boldsymbol{E}} \ne 0 \quad \text{and } \lim f = 0 \quad \text{(loading)}, \end{aligned} \tag{5.13}$$

where the yield function f is still of the form (4.11)[44]. During loading ($\hat{f} = 0$) the flow and the hardening rules have the forms (Casey 1986, Secs. III and V):

$$\dot{\boldsymbol{E}} = \gamma \boldsymbol{\varrho}, \qquad \dot{\kappa} = \gamma \lambda, \qquad (\gamma > 0), \tag{5.14}$$

where γ is a scalar function which does not require a constitutive equation and is determined from the various ingredients of the theory. The three classifications of strain-hardening characterization corresponding to (4.18a,b,c) in the elastic-plastic theory are now provided by

$$\begin{aligned} &\text{(a)}\ \ \boldsymbol{\Gamma} > 0 \quad \text{(hardening)},\\ &\text{(b)}\ \ \boldsymbol{\Gamma} < 0 \quad \text{(softening)},\\ &\text{(c)}\ \ \boldsymbol{\Gamma} = 0 \quad \text{(perfectly plastic)}, \end{aligned} \tag{5.15}$$

[43] A special case of such a variation of the function Φ was studied by Casey and Lin (1983); and, in a different context of two-dimensional strain cycling with small deformation, a plot of Φ as a function of plastic strain that characterizes strain-hardening is included in a paper of Naghdi and Nikkel (1986).

[44] In the theory of elastic-plastic materials, yield surfaces exist in both strain and stress space, but in the rigid-plastic limit the yield surface in strain space collapses to a point.

where

$$\Gamma = -\left(\dot{\lambda}\frac{\partial f}{\partial \kappa} + \boldsymbol{\varrho}\cdot\frac{\partial f}{\partial \boldsymbol{E}}\right) \tag{5.16}$$

is the "consistency" condition for rigid plastic materials ensuring that loading from a plastic state leads to another plastic state. The constitutive expressions $(5.14)_{1,2}$ and the condition (5.16) are the analogue of $(5.5)_{1,2}$ and (5.6) for rigid-plastic materials. Also, during hardening or softening the flow and the hardening rules $(5.14)_{1,2}$ can be rewritten in the forms

$$\dot{E} = \frac{\hat{f}}{\Gamma}\boldsymbol{\varrho}, \qquad \dot{\kappa} = \frac{\hat{f}}{\Gamma}\lambda \quad (\Gamma \neq 0) \tag{5.17}$$

where $\hat{f}$ is still given by $(4.17)_2$. During loading, the current value of the strain ($\boldsymbol{E} = \boldsymbol{E}_p$) has a nonzero velocity in strain space and the yield surface in stress space locally may (i) expand in a region of hardening, (ii) contract in a region of softening, or (iii) remain stationary in a region of perfectly plastic behavior. There are significant differences between the theory of rigid-plastic materials outlined between (5.13)–(5.15) and other existing formulations of rigid plasticity, including that of Hill (1962). The differences are elaborated upon in the last paragraph of Casey's (1986, p. 274) paper and need not be repeated here.

(4) The well-known Prandtl-Reuss constitutive equations for small deformation of elastic-perfectly plastic materials actually employ a strain-based flow rule even though the significance of the strain-space formulation was not yet recognized. In order to verify the truth of this statement, recall that the specific assumptions employed in the derivation of the Prandtl-Reuss relations are (a) the total strain is the sum of the elastic and plastic parts (an assumption which holds in any constitutive development of the linearized theory of elastic-plastic solids and not just Prandtl-Reuss equations), (b) the assumption of plastic incompressibility, (c) the stress response is specified by the generalized Hooke's law, (d) the yield function in stress space is specified by the von Mises yield function (in stress space) which does not depend on the mean normal of stress and is quadratic in the deviatoric components of stress, (e) the material is perfectly plastic beyond the elastic range and that the flow rule for the rate of (infinitesimal) plastic strain is linear in the deviatoric components of the stress. The specific assumptions (a) to (e) in the order stated can be recorded as

$$\text{(a)}\ \gamma_{ij} = \gamma^{e}_{ij} + \gamma^{p}_{ij}, \quad e_{ii} = e^{e}_{ii} + e^{p}_{ii}, \tag{5.18}$$

$$\text{(b)}\ e^{p}_{ii} = 0, \ \text{ and hence } \ e_{ii} = e^{e}_{ii} \ (\text{or } \bar{e} = \bar{e}^{e}), \tag{5.19}$$

$$\text{(c)}\ \tau_{ij} = 2\mu\gamma^{e}_{ij}, \quad \bar{s} = 3k\bar{e}, \tag{5.20}$$

(d) $f = \frac{1}{2}\tau_{ij}\tau_{ij} - K^2 = 0$ and
$$g = 2\mu^2\gamma^e_{ij}\gamma^e_{ij} - K^2, = 2\mu^2(\gamma_{ij} - \gamma^p_{ij})(\gamma_{ij} - \gamma^p_{ij}) - K^2, = 0 \tag{5.21}$$

(e) $$\dot{\gamma}^p_{ij} = \bar{\psi}\tau_{ij} \tag{5.22}$$

where the notations $(\gamma_{ij}, \gamma^p_{ij}, \tau_{ij})$ and $\frac{1}{3}(\bar{e}_{ii}, \bar{e}^p_{ii}, \bar{s}_{ii})$ are the rectangular Cartesian components of the tensor quantities (γ, γ_p, τ), $(\bar{e}\boldsymbol{I}, \bar{e}_p\boldsymbol{I}, \bar{s}\boldsymbol{I})$ introduced in subsection 4H (before Eq. (4.27)). In the above formulae, K^2 corresponds to the yield limit in simple shear, μ is the shear modulus introduced earlier in subsection 4H and k is the bulk modulus of elasticity. Also, the yield function g in strain space given by $(5.21)_2$ is obtained from $(5.21)_1$ with the use of (5.20) by virtue of (4.13) and the coefficient $\bar{\psi}$ in (5.22) may depend on strain and strain rate. A number of special results appropriate for Prandtl-Reuss equations follow from (5.18) to (5.21) by straightforward calculations. Thus, from the time derivatives of the yield conditions $(5.21)_{1,2}$ and the use of (5.20) we obtain

$$\begin{aligned} \dot{f} &= \tau_{ij}\dot{\tau}_{ij} = 0 \Rightarrow \gamma^e_{ij}\dot{\tau}_{ij} = 0, \\ \dot{g} &= 4\mu^2\gamma_{ij}\dot{\gamma}^e_{ij} = 0 \Rightarrow \tau_{ij}\dot{\gamma}^e_{ij} = 0. \end{aligned} \tag{5.23}$$

From the inner product of τ_{ij} and the time rate of (5.18), as well as $(5.23)_2$, follows the identity

$$\tau_{ij}\dot{\gamma}_{ij} = \tau_{ij}\dot{\gamma}^p_{ij}. \tag{5.24}$$

Also, expressions for $\hat{g}$ and $\hat{f}$ calculated from $(4.17)_{1,2}$ and $(5.21)_{1,2}$ are

$$\hat{g} = 2\mu\tau_{ij}\dot{\gamma}_{ij}, \qquad \hat{f} = \tau_{ij}\dot{\tau}_{ij}. \tag{5.25}$$

Next, substitute the flow rule (5.22) on the right-hand side of (5.24) to obtain $\tau_{ij}\dot{\gamma}_{ij} = 2K^2\bar{\psi}$ which can be solved for $\bar{\psi}$ and written in the form

$$\bar{\psi} = \frac{1}{2K^2}\tau_{ij}\dot{\gamma}_{ij} = \frac{\hat{g}}{4\mu K^2}, \tag{5.26}$$

in view of $(5.25)_1$. Finally, by considering the time derivative of $(5.20)_1$ after substituting from $(5.18)_1$ for $\gamma^e_{ij} = \gamma_{ij} - \gamma^p_{ij}$ and using $(5.26)_1$, we may obtain the following constitutive equation holding during plastic flow

$$\dot{\tau}_{ij} = 2\mu\left(\mathscr{I}_{ijkl} - \frac{1}{2K^2}\tau_{ij}\tau_{kl}\right)\dot{\gamma}_{kl}, \tag{5.27}$$

where $\mathscr{I}_{ijkl}$ are the components of the fourth order unit tensor defined following (5.11). Clearly, the results obtained between (5.23)–(5.27) are consistent with the conditions for perfectly plastic behavior in the context of strain-based formulation; in this connection, see $(4.20)_3$ and Table 1 in subsection 4F.

5C. *Restrictions on constitutive equations*

An important aspect of the development of macroscopic theories of the type under discussion involves obtaining realistic material response—in accordance with well-conceived ideas and/or experimental observations—by imposing physically plausible restrictions on the constitutive equations. For example, such restrictions may be effected by some appropriate statement (or statements) of the Second Law of thermodynamics, or by an appeal to a physically acceptable concept regarding the existence of a strain energy density in the purely mechanical theory of elastic materials (G. Green 1839).

The idea of placing some sort of restrictions on the constitutive equations of work-hardening elastic-plastic materials (e.g., the normality of plastic strain rate and convexity of yield surfaces) was originated by Drucker. Thus, with the limitation to small deformation and in a stress space setting, he advanced the fruitful idea (Drucker 1952) involving the notion that a certain work-like expression be nonnegative in a stress cycle[45]. A related postulate, again in the context of the linearized theory with small deformation and involving a "postulate of plasticity" that the integral of the stress power be nonnegative in a strain (rather than stress) cycle, was later introduced by Il'iushin (1961). As noted by Il'iushin, his postulate is less restrictive than Drucker's postulate (1952, 1959)[46]. Additional related remarks or developments in the context of infinitesimal plasticity can be found in the papers of Naghdi (1960, section 4), Drucker (1964) and Palmer, Maier and Drucker (1967).

Within the scope of the nonlinear theory, thermodynamical restrictions so far have produced rather limited results. For example, thermodynamical restrictions (to the extent that they were explored by Green and Naghdi 1965, 1966 using the Clausius-Duhem inequality) result in an expression for the stress in terms of the specific Helmholtz free energy function and a further restriction on this function in the form of an inequality involving partial derivatives of the Helmholtz free energy with respect to $\boldsymbol{E}_p$, κ and the normal to the yield surface in stress space. With this background and a desire for obtaining some systematic constitutive restrictions in the purely

[45] The restriction proposed by Drucker (1952) was stated in the context of the purely mechanical rate-independent theory of plasticity and may be summarized as follows: Consider an element of an elastic-plastic material having an existing state of stress on or inside a loading surface, to which (by an external agency) an additional set of stresses is slowly applied and slowly removed; then, in an infinitesimal cycle of application-and-removal of the added stresses, the work done by the external agency is nonnegative. An extension of this postulate to rate-dependent theory, again in the presence of small deformation, was subsequently advanced by Drucker (1959) and was called by him a "stability postulate."

[46] A close reading of Il'iushin's paper seems to suggest that his "postulate of plasticity" may have been intended as an energetic criterion for plastic flow (in conjunction with the definition of plastic strain), in contrast to Drucker's postulate which clearly was intended as a restriction on constitutive aspects of the theory.

mechanical theory, the following work assumption was proposed by Naghdi and Trapp (1975b): *The external work done on the body by surface tractions and by body forces in any sufficiently smooth spatially homogeneous cycle during the time interval* (t_0, t_f) is nonnegative, i.e.,

$$\int_{t_0}^{t_f}\left[\int_{\partial\mathscr{R}_0} {}_R\boldsymbol{t}\cdot\boldsymbol{v}\,dA+\int_{\mathscr{R}_0}\varrho_0\boldsymbol{b}\cdot\boldsymbol{v}\,dV\right]dt\ge 0 \tag{5.28}$$

for all cycles $\mathscr{C}(t_0, t_f)$, where the times t_0 and t_f designate the time of the beginning and end of the cycle. Also in (5.28), $\mathscr{R}_0$ is the region occupied by the body in the fixed reference configuration $\boldsymbol{\kappa}_0$, $\partial\mathscr{R}_0$ designates the closed boundary surface of $\mathscr{R}_0$ having the outward unit normal $\boldsymbol{N}$, dA and dV are, respectively, the elements of area and volume in $\boldsymbol{\kappa}_0$ and ${}_R\boldsymbol{t}=\boldsymbol{P}\boldsymbol{N}$ is the stress vector which acts across any surface in the current configuration $\boldsymbol{\kappa}$ but is measured per unit area of the surface in $\boldsymbol{\kappa}_0$. It should be emphasized here that given any smooth closed strain trajectory in $\mathscr{E}$, a corresponding smooth homogeneous motion can always be found such that at every particle of the elastic-plastic medium the strain calculated from the homogeneous motion equals that on the strain trajectory. Such a motion can be maintained by a suitable choice of the body force field in the consequence of the balance of linear momentum (3.3).

With the use of the expression (3.6) for mechanical power, the equations of motion (3.3) and the fact that for the homogeneous cycle associated with (5.28) the kinetic energy returns to its initial value at the end of the cycle, the work assumption (5.28) leads to the inequality (for details, see Naghdi and Trapp 1975b):

$$I=\int_{t_0}^{t_f}\boldsymbol{S}\cdot\dot{\boldsymbol{E}}\,dt\ge 0, \tag{5.29}$$

where I defines the integral in (5.29). Having obtained the inequality (5.29), we now proceed to derive both the necessary and sufficient conditions for the validity of the work inequality (5.28). However, because the derivations of the necessary conditions have evolved in the literature by different procedures and at various levels of relative simplicity, in the interest of clarity we postpone identifying the original sources of these derivations until later in this subsection.

The inequality (5.29) implies an expression for the stress $\boldsymbol{S}$ in terms of the partial derivative with respect to $\boldsymbol{E}$ of a scalar potential ψ, namely

$$\boldsymbol{S}=\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}},\qquad \psi=\hat{\psi}(\mathscr{V}). \tag{5.30}$$

With the use of the result (5.30), the integral in (5.29) can be reduced to the

following equivalent integral:

$$I = \int_{t_{0_{\text{Loading}}}}^{t_f} H(t)\,dt, \tag{5.31}$$

where the term 'Loading' below the integral signifies that the integral (5.31) has a nonzero value only during loading (or any sequence of loading) in the cycle of motion and where

$$H(t) = \left[\left(\frac{\partial\hat{\psi}}{\partial \boldsymbol{E}_p}(\tilde{\mathcal{U}}(t)) - \frac{\partial\hat{\psi}}{\partial \boldsymbol{E}_p}(\mathcal{U}(t))\right)\boldsymbol{\varrho}(\mathcal{U}) + \left(\frac{\partial\hat{\psi}}{\partial \kappa}(\tilde{\mathcal{U}}(t)) - \frac{\partial\hat{\psi}}{\partial \kappa}(\mathcal{U}(t))\right)\lambda(\mathcal{U})\right]\hat{g}(t). \tag{5.32}$$

The expression (5.31) is derived from $(5.29)_1$ by considering a smooth homogeneous cycle $\mathscr{C}(t_0, t_f)$ which begins at an interior point $\boldsymbol{E}(t_0) = \boldsymbol{E}_0$ of the loading surface $\partial\mathscr{E}$ and returns to the same value $\boldsymbol{E}_0 = \boldsymbol{E}(t_f)$ at the completion of the cycle. Also, the abbreviation $\tilde{\mathcal{U}}$ in (5.29) stands for

$$\tilde{\mathcal{U}} = \mathcal{U}|_{\boldsymbol{E}=\boldsymbol{E}_0} = (\boldsymbol{E}_0, \boldsymbol{E}_p, \kappa), \quad g(\tilde{\mathcal{U}}) \le 0, \tag{5.33}$$

and the remaining quantities in (5.32) are defined earlier in this section (see Eqs. $(4.7)_1$, (4.14), $(4.17)_1$, $(5.5)_{1,2}$).

Now the combination of the integral (5.31) and the inequality (5.29) at once results in an alternative representation of the latter inequality in the form

$$\text{integral } I \text{ defined by } (5.31) \ge 0. \tag{5.34}$$

The *inequality* (5.34) *implies the following conditions* during loading ($\hat{g} > 0$)[47]:

$$H(t)\,\hat{g} \ge 0 \quad (g(\mathcal{U}) = 0,\ g(\tilde{\mathcal{U}}) \le 0), \tag{5.35}$$

$$\hat{\boldsymbol{\sigma}} = -\gamma\frac{\partial g}{\partial \boldsymbol{E}}, \quad \gamma = \gamma(\mathcal{U}) \ge 0 \quad (g(\mathcal{U}) = 0), \tag{5.36}$$

where the scalar function H and the tensor function $\hat{\boldsymbol{\sigma}}$ are defined by (5.32) and (5.12), respectively. The constitutive restrictions (5.30), (5.35) and (5.36) are necessary conditions for the validity of the work inequality (5.28). The condition (5.30) establishes the existence of a stress potential ψ for the elastic-plastic materials under consideration. The condition (5.35) represents an additional constitutive restriction on the first partial derivatives of ψ which is also related to convexity of yield surfaces. Finally, the result (5.36) may be referred to as the normality condition—it represents the fact that the tensor $\hat{\boldsymbol{\sigma}}$ is directed along the normal to the yield surface in strain space.

[47] The details are not included here but will be found in the references that will be cited presently.

The condition (5.30) was first given by Casey and Naghdi (1984a), although it was assumed in the paper of Naghdi and Trapp (1975b, section 2). The restriction (5.35) was first derived by Casey and Tseng (1984) from (5.29); and, as was noted by them, it is possible to recast this inequality in a variety of forms. The normality condition (5.36) was first derived by Naghdi and Trapp (1975b) from (5.28), although a much simpler proof was supplied later by Casey (1984). Still, a more direct derivation of the necessary conditions (5.35)–(5.36) is included in a recent paper by Lin and Naghdi (1989), where (5.30) and (5.35) have been also identified as the conditions sufficient for the validity of the work inequality (5.28). The fact that the normality condition is only necessary but not sufficient may not come as a surprise. There are certainly examples, both in the case of metals and geological materials, where the normality does not hold and one must return to the more basic constitutive results (often referred to as nonassociative flow rules) in the forms (5.5)–(5.3) in the absence of[48] $\varkappa$.

In a recent discussion of a paper by Carroll (1987) on finite inelastic deformation, Hill and Rice (1987) have claimed that they have also derived a normality condition corresponding to the result (5.36). This claim can be disputed for the following reasons: The derivation by Hill and Rice (1973) starts with a generalized version (to finite deformation) of Il'iushin's integral of the stress power *and* utilizes infinitesimal quantities of different orders, as has been pointed out also in Carroll's response to their discussion[49]. Since the steps between an inequality involving the integral of the stress power and a result such as (5.36) are purely mathematical, the use of "infinitesimals of different order" hardly qualifies as a proof. In addition, the development by Hill and Rice (1973) begins with an integral of the stress power which is not the same as the work inequality (5.28); the latter is a physically motivated dynamical restriction which is valid for *every* smooth homogeneous cycle of motion[50], whereas the former is not.

6. Thermal effects. Rate-dependent behavior

As remarked in section 1, many of the basic features of the rate-independent behavior are also exhibited in closely related regimes such as thermoplasticity and viscoplasticity. We, therefore, take advantage of the

[48] An example of nonassociative flow rule (i.e., when normality does not hold) in the case of metals arises in connection with the phenomenon of the strength-differential effect which has been discussed in the literature from both theoretical and experimental points of view (see Drucker 1973, Spitzig *et al.* 1975, Casey and Jahedmotlagh 1984, Casey and Sullivan 1985).

[49] Carroll's response appears under Author's Closure in the Discussion of Hill and Rice (1987).

[50] See, in this connection, the last two sentences of the paragraph containing (5.28).

detailed background material pertaining to various constitutive ingredients[51] already discussed in section 4 and will be brief in the introduction of the parallel features in this section. Indeed, as will become evident presently, many of the basic features associated with the rate-independent plastic deformation in thermoplasticity—aside from an extra dependence on the temperature—remarkably parallel the same format as in the rate-independent mechanical theory.

6A. *Thermal effects in plasticity*

The inclusion of thermal effects necessarily requires consideration of a complete thermomechanical theory and will be dealt with here only in the context of the rate-independent theory of elastic-plastic materials. It may be recalled that in the development of any thermomechanical theory of material behavior, the usual kinematics and kinetics of the purely mechanical theory (summarized in sections 2 and 3) must be supplemented by a number of thermal variables[52], i.e., an absolute (positive) temperature $\theta = \theta(X, t)$, temperature gradient $g_R = \partial\theta/\partial X$, the specific external rate of supply of heat r, the heat flux vector q_R measured per unit area in the reference configuration, the specific entropy η, the specific internal energy ε and the specific Helmholtz free energy defined by $\psi = \varepsilon - \theta\eta$. In addition to the balance laws of the purely mechanical theory, we also have a balance of energy (or the First Law of thermodynamics) which we record here in the local form[53]

$$\varrho_0 \dot{r} - \varrho\dot{\varepsilon} - \operatorname{Div} q_R + P = 0, \tag{6.1}$$

or in terms of the specific Helmholtz free energy as

$$\varrho_0 r - \varrho_0(\dot{\psi} + \eta\dot{\theta} + \theta\dot{\eta}) - \operatorname{Div} q_R + P = 0. \tag{6.2}$$

The "Div" operator in (6.1)–(6.2) has been previously defined following (3.3) and the mechanical power P is given by (3.6).

In the presence of thermal effects, the structure of the constitutive assumptions for the stress response, the rate of plastic strain $\dot{E}_p$ and the rate of hardening $\dot{\kappa}$—apart from an extra dependence on temperature—remain the same as the corresponding assumptions in the purely mechanical theory discussed in sections 4 and 5. Keeping this in mind, we observe that nearly all developments in the literature of thermoplasticity during the past two

[51] By these we have in mind such constitutive ingredients as the concept of yield, yield surfaces in strain and stress spaces, the loading criteria, strain trajectory in strain space and the corresponding stress trajectory in stress space and so on.
[52] All these thermal variables are functions of position and time, although this dependence is emphasized in the text only in the case of absolute temperature.
[53] Equations (6.1) and (6.2) represent the Lagrangian forms of the balance of energy. The corresponding Eulerian forms are not needed in the present discussion and can be readily found in the literature (see, for example, section 3 of Green and Naghdi 1965).

decades are stress-based and often employ an Eulerian representation if the discussions and derivations are carried out in the context of finite deformation. Also, nearly all the papers which include thermal effects, regardless of whether the deformation is assumed to be finite or infinitesimal, after introducing a set of constitutive equations for the thermal variables (such as the specific Helmholtz free energy, the specific entropy and the heat flux vector) appeal to a statement of the Second Law of thermodynamics usually in the form of the Clausius-Duhem inequality. Setting aside temporarily the thermodynamical aspects, we briefly comment here on the inadequacies of the mechanical part of the constitutive structure in a few representative papers on thermoplasticity.

In a frequently cited paper, Mandel (1973) addresses the general formulation of an elastic-plastic theory in the presence of thermal effects. After some discussion of the so-called hidden and internal variables[54], he mentions that the theory may be formulated in terms of an Eulerian description (which he also calls "present stressed configuration") or in terms of a "present released configuration" (Mandel 1973, p. 285, subsection 15.3.1) and states that "it is necessary to determine in some way the orientation of the present (stressed or released) configuration, so that an orientation variable must be added to the state variables." He then represents the "orientation variable" by an "orthonormal triad," which he calls a director triad[55]. Subsequently, with the use of a multiplicative decomposition of the type (4.1) and its rate involving the velocity gradient $\boldsymbol{L}$ (see his Eqs. (15.4)–(15.6)), Mandel's analysis culminates in his expressions representing constitutive equations (Mandel 1973, p. 296) for an "elastic rate of deformation" denoted by $\boldsymbol{D}_e$ and for a "plastic rate of deformation." These latter constitutive ingredients in Mandel's paper, apart from the fact that they include the effect of his "orientation variable," are variants of the decomposition (4.23) and a constitutive equation of the type (4.24); and, at the very least, are subject to the same criticisms as those already discussed in subsections 4A and 4G. Also, it should be noted that Mandel's choice of an "orientation variable" is much too restrictive. Indeed, since this variable is represented by an *orthonormal* set of directors, then (i) there would be no change in the magnitude of the directors and (ii) the angles between the directors are fixed and will remain the same during the entire motion.

In the same vein, the structure of the constitutive equations for the flow and hardening rules in many of the recent papers on thermoplasticity are

[54] The terminology of hidden or internal variable is sometimes used in the literature to identify those variables that are not directly observable. A variable of this kind requires an evolution equation; and, in the context of inelastic behavior, is specified by a rate-type constitutive equation such as those for $\hat{\boldsymbol{E}}$, $\boldsymbol{\alpha}$ and κ.

[55] Presumably, the "orientation variable" is intended to account for the effect of orientation of a microelement (such as a crystal) in the macroscopic theory. Mandel's paper also contains some discussion on micromechanics, but we defer comments on this aspect of his paper until section 8.

subject to the same type of criticism already discussed in subsections 4G and 4H regarding the decomposition of the rate of deformation $\boldsymbol{D}$ and the spin tensor $\boldsymbol{W}$ into their respective elastic and plastic parts ($\boldsymbol{D}_e$, $\boldsymbol{D}_p$ and $\boldsymbol{W}_e$, $\boldsymbol{W}_p$), together with a constitutive equation of the type (4.24) for $\boldsymbol{D}_p$. Among these mention may be made of the papers by Anand (1985) and by Anand and Lush (1987), which contain additional references to earlier similar developments. In another typical approach to thermoplasticity, Eisenberg, Lee and Phillips (1977) employ internal variables which are not explicitly defined and use an additive decomposition for strain[56]. In addition, as in many similar papers on the subject that utilize internal variables, Eisenberg *et al.* (1977) also confine attention to a restricted class of constitutive equations.

In the preceding two paragraphs, we have briefly pointed out some of the shortcomings of the mechanical aspects of representative papers on thermoplasticity. Keeping this in mind, we now proceed to the presentation of an outline of a strain-based formulation of plasticity extended to include thermal effects. Thus, having already argued for the primacy of strain-space formulation, we regard the set of quantities

$$\{\boldsymbol{E}, \theta, \mathscr{W}\}, \tag{6.3}$$

with the abbreviation $\mathscr{W}$ standing for the set of variables defined by (5.4). Then, in parallel with the development of strain-space formulation of the purely mechanical theory, we admit the existence of a scalar-valued yield function g in the strain-temperature space in the form

$$g(\boldsymbol{E}, \theta, \mathscr{W}) = 0. \tag{6.4}$$

For fixed values of $\mathscr{W}$, the equation (6.4) represents a closed orientable hypersurface of dimension six enclosing an open region in the seven-dimensional strain-temperature space[57]. The function g on the left-hand side of (6.4) is chosen such that $g(\boldsymbol{E}, \theta, \mathscr{W}) < 0$ for all points in the interior of the yield surface in strain-temperature space. Next, we introduce a constitutive equation for the stress response in the form[58]

$$\boldsymbol{S} = \text{function of the variables (6.3)} \tag{6.5}$$

[56] These authors write $e_{ij} = e'_{ij} + e''_{ij}$ and go on to assume that inside the yield surface in stress-temperature space the rate of e''_{ij} is a linear function of the rate of stress and the rate of temperature without an explicit identification of e'_{ij} as elastic strain, or even a mention of whether the strain is finite or infinitesimal.

[57] The yield condition g in (6.4) corresponds to (4.14) in the mechanical theory; and, similarly, the yield condition (6.8) given below corresponds to (4.11) in the mechanical theory. Yield surfaces of the type (6.8) have been shown experimentally to exist in stress-temperature space and have been reported by Phillips (1974) on the basis of data obtained from small deformation of thin-walled specimens of pure aluminum subjected to combined tension-compression and torsion at various temperatures well above the room temperature. For additional remarks on these experiments, see subsection 7A.

[58] Similar to (4.6), a constitutive equation of the form (6.5) can also be expressed in terms of an equivalent set of variables $(\boldsymbol{E} - \boldsymbol{E}_p, \theta, \mathscr{W})$.

and assume that for fixed values of $(\theta, \mathscr{W})$ this can be inverted to yield

$$E = \text{function of } \{S, \theta, \mathscr{W}\}. \tag{6.6}$$

Once a constitutive equation of the type (6.5) and its inverse (6.6) is adopted, then an expression for the yield function f in stress-temperature space can be easily found through the relation

$$g(E, \theta, \mathscr{W}) = g(\hat{E}(S, \theta, \mathscr{W}), \theta, \mathscr{W}) = f(S, \theta, \mathscr{W}). \tag{6.7}$$

Again, for fixed values of $\mathscr{W}$, the equation

$$f(S, \theta, \mathscr{W}) = 0 \tag{6.8}$$

represents a closed hypersurface of dimension six which encloses an open region in stress-temperature space and has the same geometrical properties as the yield surface (6.4) in the strain-temperature space. In parallel to the development of loading criteria in subsection 4E, we take the yield (or loading) function and the loading criteria in strain-temperature space as primary and define the quantities $\hat{g}$ and $\hat{f}$ by[59]

$$\hat{g} = \frac{\partial g}{\partial E} \cdot \dot{E} + \frac{\partial g}{\partial \theta} \dot{\theta}, \qquad \hat{f} = \frac{\partial f}{\partial S} \cdot \dot{S} + \frac{\partial f}{\partial \theta} \dot{\theta}. \tag{6.9}$$

Then, analogously to (4.18), the loading criteria of the strain-temperature-space formulation are defined to be

(a) $g < 0$, (*thermoelastic state*),

(b) $g = 0$, $\hat{g} < 0$ (unloading from a thermoelastic-plastic state),

(c) $g = 0$, $\hat{g} = 0$ (neutral loading),

(d) $g = 0$, $\hat{g} > 0$ (loading), (6.10)

and the derived conditions in stress-temperature space corresponding to $(6.10)_{1,2,3,4}$ are: $f < 0$ (thermoelastic state); $f = 0$, $\hat{f} < 0$ (during unloading); and $f = 0$, $\hat{f} = 0$ (during neutral loading). Also, the loading conditions in stress-temperature space that can exist in conjunction with (6.10) permit a classification of strain-hardening characterization identical to (4.20) but with Φ now dependent on the variables (6.3), i.e.,

$$\Phi = \Phi(E, \theta, \mathscr{W}) = \frac{\hat{f}}{\hat{g}}. \tag{6.11}$$

Moreover, the relations between the loading criteria in strain-temperature space and associated conditions in stress-temperature space have the same

[59] The quantities defined by $(6.9)_{1,2}$ represent the counterparts of $\hat{g}$ and $\hat{f}$ defined by $(4.17)_{1,2}$ in the mechanical theory.

format as those summarized in Table 1 (of subsection 4F). Just as in the mechanical theory, it should be evident that the formulation of the thermomechanical theory in the strain-temperature and the stress-temperature spaces are not equivalent.

We now discuss the nature of the remaining constitutive ingredients of the thermomechanical theory[60]. The constitutive equations for the rate of plastic strain $\dot{\boldsymbol{E}}_p$ and for $\dot{\kappa}$ and $\dot{\boldsymbol{\alpha}}$ during loading will have the same form as those in the mechanical theory but their coefficient functions in $(5.5)_{1,2,3}$ now depend on the variables (6.3) instead of $\mathscr{V}$ defined by (5.2). As in the development of the constitutive results of the purely mechanical theory during loading, it can be readily verified that the constitutive assumptions of the forms $(4.22)_1$, $(4.26)_1$ and (4.28), as well as the "consistency" condition, again reduce to the forms $(5.5)_{1,2,3}$ and (5.6), except that g and the various coefficients are now functions of the variables (6.3).

In order to complete the discussion of our constitutive assumptions, we introduce the constitutive equations for ψ and η in the form

$$\psi = \hat{\psi}(\boldsymbol{E}, \theta, \mathscr{W}), \quad \eta = \hat{\eta}(\boldsymbol{E}, \theta, \mathscr{W}), \tag{6.12}$$

and specify a constitutive equation for $\boldsymbol{q}_R$ with the heat flux response being dependent on the variables (6.3) and the temperature gradient $\boldsymbol{g}_R$ defined in the opening paragraph of this subsection.

At this stage of the development of any thermomechanical theory, it is desirable to place some restrictions on the various response functions in the theory such as those which occur in (6.5) and (6.12); and this is effected by an appeal to ideas arising from the Second Law of thermodynamics. In most of the current literature this is carried out with the help of the Clausius-Duhem or similar inequalities, where often the concept of entropy first appears. An alternative procedure—motivated by the structure of balance of energy in the special case of inviscid fluids—has been developed by Green and Naghdi (1977, 1978a) where an entropy balance is postulated in addition to other balance principles[61]. It then follows that the local equation for the balance of entropy for the special case of an inviscid fluid (but *not* for other materials) has the same form as the energy equation (6.1) or (6.2), and hence is not an independent equation. Further, substitution of the local equation for the balance of entropy and the equations of motion (3.3) into the balance of energy (6.2) results in a *reduced* energy equation, which is then regarded as an identity for all thermomechanical processes. For details of this development, we refer the reader to Green and Naghdi (1977) but

[60] A discussion of constitutive equations of the thermomechanical theory in the context of strain-space formulation is contained in a paper of Green and Naghdi (1978b).

[61] The nature of thermodynamical restrictions discussed in these papers can be further supplemented by a broader set of restrictions which readily follow from the Second Law statements discussed in a paper of Green and Naghdi (1984) after suppressing the electromagnetic effects.

only note here that the reduced energy equation may be expressed in the form

$$-\varrho_0(\dot{\psi} + \eta\dot{\theta}) + P - \varrho_0\theta\xi - \frac{1}{\theta}\boldsymbol{q}_R \cdot \boldsymbol{g}_R = 0, \tag{6.13}$$

where the variable ξ in (6.13) represents the specific internal rate of production of entropy which requires a constitutive equation that depends on the variables (6.3), the rate quantities $\dot{\boldsymbol{E}}_p$, $\dot{\kappa}$, $\dot{\boldsymbol{\alpha}}$ and the temperature gradient $\boldsymbol{g}_R$ (Green and Naghdi 1978b).

We now proceed to indicate the nature of the thermodynamical restrictions for an elastic-plastic material and introduce the various constitutive assumptions into the reduced energy equation (6.13) which must be satisfied identically for every motion. Several restrictions then follow from (6.13) and are given by[62]

$$\boldsymbol{S} = \varrho_0 \frac{\partial\hat{\psi}}{\partial \boldsymbol{E}}, \qquad \eta = -\frac{\partial\hat{\psi}}{\partial\theta}, \tag{6.14}$$

and an equation of the form

$$\varrho_0 \frac{\partial\hat{\psi}}{\partial \boldsymbol{E}_p} \cdot \dot{\boldsymbol{E}}_p + \varrho_0 \frac{\partial\hat{\psi}}{\partial\kappa}\dot{\kappa} + \frac{\partial\hat{\psi}}{\partial\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} + \varrho_0\theta\xi + \frac{1}{\theta}\boldsymbol{q}_R \cdot \boldsymbol{g}_R = 0. \tag{6.15}$$

The equation for the local entropy balance (not displayed here) can be reduced to

$$\varrho_0 r - \operatorname{Div} \boldsymbol{q}_R + \frac{1}{\theta}\boldsymbol{q}_R \cdot \boldsymbol{g}_R + \varrho_0\theta\xi = \varrho_0\theta\dot{\eta} \tag{6.16}$$

and is an equation for the determination of the temperature field, once the constitutive equations for ψ, ξ and $\boldsymbol{q}_R$ are known. Further restrictions can be placed on the constitutive equations with the help of appropriate statements of the Second Law in a manner discussed by Green and Naghdi (1984) but so far in this development no appeal has been made to a Second Law of thermodynamics[63].

To illustrate how further simplifications can be introduced into the basic development of the theory, we first recall that the various constitutive ingredients can alternatively be expressed in terms of the following equivalent set of variables

$$(\boldsymbol{E} - \boldsymbol{E}_p, \theta, \mathscr{W}), \tag{6.17}$$

[62] With the stipulation that the reduced energy equation be satisfied identically for all motions, in the procedure of Green and Naghdi (1977) the local balance of entropy (rather than the energy equation) is used as the equation for the determination of the temperature field.

[63] It is perhaps of interest to note that the results $(6.14)_{1,2}$ and (6.15), but not (6.16), are of the same form as those that can be deduced with the use of the Clausius-Duhem inequality referred to earlier in this subsection.

with the abbreviation $\mathscr{W}$ defined by (5.4). Then, assuming that the specific Helmholtz free energy (but not necessarily the other dependent variable) does not depend explicitly on[64] $\mathscr{W}$, i.e.,

$$\psi = \bar{\psi}(\boldsymbol{E} - \boldsymbol{E}_p, \theta), \tag{6.18}$$

by the procedure that led previously to (6.14)–(6.15) and the fact that $\partial\bar{\psi}/\partial\boldsymbol{E}_p = \partial\bar{\psi}/\partial\kappa = 0$, we now obtain

$$\boldsymbol{S} = \varrho_0 \frac{\partial\bar{\psi}}{\partial(\boldsymbol{E} - \boldsymbol{E}_p)}, \qquad \eta = -\frac{\partial\bar{\psi}}{\partial\theta}, \tag{6.19}$$

and the restriction

$$-\boldsymbol{S}\cdot\dot{\boldsymbol{E}}_p + \varrho_0\theta\xi + \frac{1}{\theta}\boldsymbol{q}_R\cdot\boldsymbol{g}_R = 0. \tag{6.20}$$

The equation for the determination of temperature is again given by (6.16), but can now be simplified in view of (6.20). The special results (6.19)–(6.20) and (6.16) are particularly of interest in regard to small deformation of elastic-plastic materials with $\bar{\psi}$ taken as a quadratic function of $(\boldsymbol{E} - \boldsymbol{E}_p, \theta)$, while both ξ and $\boldsymbol{q}_R$ depend linearly on temperature gradient with coefficient functions which may be dependent on (6.17). Additional restrictions can be imposed on $\boldsymbol{q}_R$ with the use of a Second Law statement, but we do not pursue the matter further.

6B. *Viscoplasticity*

While a general study of rate-dependent theory of inelastic behavior of materials in the context of a purely mechanical theory is intrinsically of definite interest, such rate-dependent behavior of materials is frequently observed in conjunction with deformations at elevated temperatures. In this subsection, however, we discuss mainly the purely mechanical theory of a class of rate-dependent behavior of materials which is often categorized as elastic-viscoplastic or simply referred to as viscoplasticity.

Theoretical developments pertaining to rate-dependence of most structural materials at elevated temperatures, which also involves permanent (or plastic) deformation of the type generally admitted in the construction of rate-independent theories, have been largely confined to small deformation, although some recent studies have also been carried out in the presence of finite deformation. Historically, the existing literature on the subject may be grouped into two categories depending on whether or not a yield function—and hence also a yield condition—is admitted as part of the constitutive ingredients of the theory.

[64] This assumption corresponds to the special form (4.8) in the purely mechanical theory.

In a category that does not admit the existence of a yield function, mention should be made of the papers of Bodner and Partom (1975) and Stouffer and Bodner (1979), Lee and Zaverl (1978, 1979), Liu and Krempl (1979), among others. The paper of Bodner and Partom (1975) appears to strive for the construction of a finite rate-dependent theory and begins with the decomposition of the form (4.21). Their development, so far as finite deformation is concerned, is subject to the same type of criticism discussed in subsection 4G (following Eq. (4.22))[65]. Apart from this, the work of Bodner and Partom (1975) and other papers cited in this paragraph represent efforts towards the development of appropriate constitutive equations for small deformation of elastic-viscoplastic materials. In all of these studies, since a yield condition is not admitted, there is no separation between the elastic and the inelastic parts of deformation and the *model* in question necessarily has the feature that the inelastic part of the deformation can occur corresponding to any stress level no matter how small. A related development by Valanis (1971, 1975), which again does not admit a yield function (or yield condition), is a functional type theory known as the endochronic theory of plasticity. This work has been reviewed in some detail by Rivlin (1981a); in this connection, see also the rebuttal by Valanis (1981) and a further response on the subject by Rivlin (1981b).

In contrast to the developments outlined in the preceding paragraph, there is another category of papers on the subject of viscoplasticity with small deformation that admits the yield condition but the issue involving loading criteria either is not addressed in these papers or is discussed only vaguely. Among the papers in this category, we mention the work of Perzyna (1963, 1966) and the papers of Phillips and Wu (1973), Chaboche (1977) and Eisenberg and Yen (1981). In this category, the inelastic deformation is regarded to occur beyond the initial yield and is comprised of permanent (or plastic) and rate-sensitive (or viscous) parts. A modified version of Chaboche's (1977) procedure is discussed in a recent paper by Eftis, Abdel-Kader and Jones (1989), which also provides a comparison between the predictions of the modified Chaboche model and the Bodner-Partom constitutive equations, as well as with available experimental results for the uniaxial behavior of[66] INCONEL 718 at 1200°F for cyclic loading. While such a comparison may be interesting, it does not provide a crucial basis for distinguishing between different theories and their predictions. In fact, there seems to be several such comparisons in the literature between various developments in viscoplasticity but the comparisons are inconclusive.

[65] A generalization of the work of the theory of Bodner and Partom (1975) in the presence of finite deformation but still without employing a yield condition has been discussed in several recent papers by Rubin (1986, 1987a,b).

[66] This abbreviation refers to a nickel based superalloy. For a more detailed description, see Eftis *et al.* (1989, Sec. IV), where original references for the experimental data in cyclic tests for determination of the mechanical properties of INCONEL 718 at 1200°F can be found.

The existing literature in the theory of elastic-viscoplastic materials, which are mainly stress-based, represent largely *ad hoc* developments even in the presence of small deformation. The inadequacy of a stress-based formulation, addressed in some detail for the rate-independent theory, equally applies to the case of rate-dependent materials. In view of this, it is desirable to spell out here the ground rules for a satisfactory theoretical development of the subject. Indeed, as was pointed out previously in the context of a finitely deforming rate-dependent theory (Naghdi 1984b,c), any idealized elastic-viscoplastic model should at least accommodate the following features: (1) For sufficiently low strain rates (i.e., as the strain rate tends to zero), the rate-dependent response of the material should approach that of the rate-independent theory with the yield function (and hence also the yield condition) as one of its constitutive ingredients; (2) it should allow for a suitable definition of plastic strain (for the present at least for small deformation); (3) the constitutive model should be capable of describing the rate-sensitive material response during both loading and unloading; and (4) the stress constitutive equation during loading should accommodate the response of the medium both in the elastic range and the rate-dependent response during loading beyond the elastic limit (or initial yield). It should be noted that the above requirements are in conformity with the view that any hierarchical theory (such as an elastic-viscoplastic theory) should include a lower hierarchy (e.g., rate-independent elastic-plastic theory or ordinary nonlinear elasticity) as a special case.

We close this subsection by calling attention to an idealized model described in terms of a strain-based Lagrangian formulation of an elastic-viscoplastic material which admits: (i) a yield condition in the form (4.14) and the loading criteria (4.18) together with a yield condition in the form (4.11) and the loading conditions described in subsection 4F [see the two paragraphs preceding that which includes (4.21)], as well as the strain-hardening characterization in terms of Φ defined by (4.19); (ii) constitutive equations for the rate of plastic strain $\dot{\boldsymbol{E}}_p$ and for $\dot{\kappa}$ and $\dot{\boldsymbol{\alpha}}$ during loading in the forms $(4.22)_1$, $(4.26)_1$ and (4.28); and (iii) a constitutive equation for the stress in the form (Naghdi 1984b):

$$\boldsymbol{S} = \begin{cases} {}_1\boldsymbol{S} + \mathscr{S}\dot{\boldsymbol{E}}_p & \text{during loading when } g = 0,\ \hat{g} > 0, \quad \text{(a)} \\ {}_1\boldsymbol{S} & \text{during unloading and neutral,} \quad \text{(b)} \\ & \text{loading when } g = 0,\ \hat{g} \leq 0, \end{cases} \tag{6.21}$$

where the second order tensor ${}_1\boldsymbol{S}$ and the fourth order tensor $\mathscr{S}$ depend on the variables $\mathscr{U}$ defined by $(5.2)_1$, i.e.,

$${}_1\boldsymbol{S} = {}_1\boldsymbol{S}(\mathscr{U}), \qquad \mathscr{S} = \mathscr{S}(\mathscr{U}), \tag{6.22}$$

and satisfy obvious symmetries. The stress response characterized by (6.21) and (6.22) is linear in the strain rate $\dot{E}$ through the constitutive equation of the form $(4.22)_1$ during loading. As elaborated in Naghdi (1984b), the second part of the stress response during loading in this idealized model, namely $\mathscr{S}\dot{E}_p$, represents a jump in the stress S and will assume different values depending on the rate of strain with which loading is taking place (in this connection see Figs. 1 and 2 of Naghdi 1984b). Restrictions on the constitutive equations of this idealized model can also be obtained with the use of the work inequality (5.28) of subsection 5C (see Naghdi 1984b), but we do not discuss this development here.

It should be evident from the foregoing discussion of viscoplasticity that even though the subject is of considerable interest in applications and has received increasing attention during the past three decades, as a whole the current state of the subject is unsatisfactory. However, one would hope that the combination of decisive new experiments and further work on the foundations of the subject would lead to a better state of understanding of rate-dependent behavior of materials in the near future.

7. Experimental and computational aspects

The conduct of fundamental and crucial experiments is of utmost importance to the theoretical development of any scientific field of research, especially one as complex as the inelastic behavior of materials. In addition, given the current and future computational capabilities, well-conceived numerical simulations will supplement the knowledge gained from the experimental data and will enormously enhance our understanding of the field. It may be emphasized that all three directions of research, i.e., theoretical, experimental and computational, are essential in fostering further advancement of finite plasticity. Thus, in this section we discuss separately experimental interpretations and computational potential in light of the theoretical developments summarized in sections 5 and 6. Although the discussions that follow are intended to emphasize finite deformation, they are also of interest in the context of infinitesimal plasticity.

7A. *Some suggestions for experiments in plasticity*

Given the present state of our understanding of inelastic behavior of materials in general and finite plasticity in particular, it is not completely clear as to which experimental directions will be fruitful and which ones will not be. Despite this, it is still desirable to sketch a broad outline of an experimental program of research in finite plasticity. Such an experimental program is essential in the further development of plasticity theory,

especially because of the large number of competing theoretical developments that now exist and continue to flourish in different directions and on the basis of a variety of viewpoints. Moreover, such a program of research should supply raw data in the context of both strain and stress spaces. It should be noted that in the past, experimental data in plasticity have been reported mainly relative to stress space and almost exclusively for small deformation only. Such partial results cannot possibly reveal the complete scope of material behavior in the full elastic-plastic range.

In line with the spirit of this review article, our discussion of the experimental topics will be largely confined to those aspects which especially bear on the rate-independent behavior of elastic-plastic materials. However, because of the important role that must necessarily be played by future experiments, we indicate here the nature of a much broader scope of experimental topics. Thus, before embarking on a detailed discussion in this subsection, we provide below a list of topics of current interest for an effective experimental program:

(a) Determination of yield (or loading) surfaces and their geometric features, e.g., their shape, size, normals to the surfaces, etc., in both strain and stress spaces and for finite deformation.

(b) Evolution of yield surfaces in both strain and stress spaces during finite motion and a general study of strain-hardening, including direct measurement of the function Φ defined by (4.17).

(c) Identification of the relationship between the rate of stress $\dot{\boldsymbol{S}}$ and the rate of strain $\dot{\boldsymbol{E}}$ by specifying a strain trajectory and measuring the relevant coefficients (such an experiment can be carried out independently of those for yield surfaces).

(d) Experiments for yield surfaces and their evolution, similar to those mentioned in (a) and (b) above, but carried out at elevated temperatures.

(e) Experiments in viscoplasticity which could test the merits of some idealized concepts in the further development of a theory for rate-dependent behavior of materials such as the jump in stress response associated with loading at different strain rates (see Naghdi 1984b, Fig. 1).

(f) Experiments in metal-forming and extrusion processes. (Existing data in this area are rather crude and have been obtained largely for calculating force resultants rather than stresses and strains.)

(g) Wave propagation and penetration both in the context of rate-independent and rate-dependent behavior.

(h) Experimental aspects of microstructural effects and crystal plasticity, especially in the presence of finite deformation[67].

[67] Although a theoretical discussion of microstructural effects and crystal plasticity is not presented until the next section, it is felt that its inclusion is desirable.

(i) Experimental studies for polymeric and geological materials, especially of the type noted under the topics (a)–(e) above.

In what follows, we elaborate on the list of topics (a)–(e), not necessarily in the order listed; and, with reference to past experiments, frequently combine the discussions of the background information pertinent to two or more topics (such as (a) and parts of (d)) in the same paragraphs. Consider now a typical experiment which would involve at least two or three nonzero components of (nearly) homogeneous strains as, for example, in the case of experiments involving combined tension-compression and torsion tests of thin tubular specimens. In such experiments, it is usual to report the measured data, such as the loci of yield surfaces, only in stress space; and this is effected by delineating the location of the new yield point in the space of shear stress and axial stress. But since the corresponding values of the strains are also known to the experimenter, it should not be difficult to provide the relevant experimental data also for the yield surfaces in strain space.

For a variety of metallic materials (such as mild steel, copper and aluminum alloys) and with the use of thin-walled specimens subjected to a two-dimensional state of stress resulting from small homogeneous deformations, considerable experimental information pertinent to the shape and other features of the initial and subsequent yield surfaces in stress space have been accumulated over the years[68]. In the absence of thermal effects, the first experiment of this kind on the shape of the initial yield surfaces was conducted by Taylor and Quinney (1931) whose data were confined to the first quadrant in the plane of shear stress-axial stress. Experimental studies of initial and subsequent yield surfaces were carried out later by Naghdi, Essenburg and Koff (1958), Ivey (1961) and Bertsch and Findley (1962), among others. Corresponding experiments on the shape of the initial and subsequent yield surfaces at elevated temperatures have been reported in the papers of Phillips, Liu and Justusson (1972), Phillips and Kasper (1973) and a detailed review of this aspect of the subject has been given by Phillips (1974). Remembering that the experimental data on the shape of yield surfaces reported under isothermal conditions (see, e.g., Taylor and Quinney 1931, Naghdi *et al.* 1958, Ivey 1961) correspond to the nearly elliptical curves in the two-dimensional stress space, it is interesting that the experimental data at elevated temperatures furnished by Phillips and co-workers seem to repre-

[68] Such two-dimensional states of stress can be maintained in combined tension-compression and torsion or combined internal pressure and torsion experiments. In connection with the experimental papers on the subject, it should be remembered that the guidelines for the interpretation of the measured data in such experiments were necessarily based on the state of the theory of plasticity which existed at the time.

sent cross sections of a truncated cone in a three-dimensional stress-temperature space; see Figs. 5, 9 and 20–22 of Phillips (1974)[69].

In any experimental program of the type discussed in the preceding two paragraphs for the determination of the initial yield surface, it is necessary to adopt a procedure on how to identify the first occurrence of plastic strain. A similar procedure must be used on reloading from an existing state in the elastic region (enclosed by the yield surface) for the determination of subsequent yield surfaces. Not all experimental results reported in the literature (including those cited in this subsection) utilize the same procedure. Often, on the assumption that the initial elastic region is delineated by the porportional limit, deviation of the uniaxial stress-strain curve from linearity is regarded as coinciding with the first appearance of plastic strain. Another commonly adopted procedure (the offset method) regards yielding to have occurred once a preassigned small plastic strain has been accumulated. The use of such different procedures has been widely noted in the literature and is also discussed by Phillips (1974). In particular, Williams and Svensson (1970), Helling and Canova (1985) and Stout, Martin, Helling and Canova (1985) have examined initial yield surfaces in combined tension and torsion determined by different values of offset (permanent) strain. The influence of such different procedures on the shape of yield surface, as well as other features such as the absence of cross-effect[70] in combined tension and torsion tests (Naghdi *et al.* 1958, Ivey 1961) has been reported for some materials. However, regardless of the metallic material used in the experiments, it is generally accepted that at room temperature the initial yield surface (in combined tension-torsion-internal pressure experiments) lies between the von Mises and Tresca yield conditions and is frequently closer to the von Mises yield condition. Evidently, based on the experimental data of Phillips and co-workers, the same conclusion holds in the case of initial yield surfaces at elevated temperatures.

Again, with reference to experiments on the initial and subsequent yield surfaces in stress space, mention should be made of past studies on the existence of corners on yield surfaces. Among such studies we cite the papers of Naghdi, Rowley and Beadle (1955), Naghdi, Essenburg and Koff (1958), Phillips (1960), Phillips and Gray (1961) and Hecker (1972, 1976). In all these experiments, thin-walled tubular specimens were subjected to biaxial "zig-zag" loading paths in stress space and in the majority of the

[69] Throughout this article we have used the terminologies of "yield" and "loading" function and hence also yield and loading surfaces synonymously. However, in the recent literature some authors (e.g., Phillips 1974 and other papers of Phillips and co-workers) while making reference to an initial yield surface have distinguished between subsequent yield surfaces and loading surfaces in the spirit of a two-surface model of Phillips and Sierakowski (1965). Comments on the use of such multi-loading surfaces were made previously in subsection 4E.

[70] The absence of "cross-effect" in combined tension and torsion tests refers to the observation that under a prestress in torsion there is no change in the size of the yield surface along the tensile stress axis.

cases the relationship between the direction of the applied stress increment (or stress rate) and the direction of the resulting plastic strain increment (or plastic strain rate) has indicated the existence of corner or pointed vertex (rather than only a localized region of high curvature) at the loading point (see Fig. 1 of Naghdi *et al.* 1955). Even in the latest study on the subject (Hecker 1972) the experimental results are inconclusive. There seems to be some dichotomy between the direct yield surface measurements on the one hand (which indicate smooth yield surfaces) and the "zig-zag" loading experiments on the other hand (which indicate corners). It is certainly of interest to clarify the issue. Given today's laboratory instrumentation, it appears that the existence of corners can now be directly tested by determining whether or not a unqiue normal exists at a given material point on the initial and subsequent yield surfaces. In the event that such experiments unambiguously establish the absence of sharp corners, it would still be of interest to clarify the results of earlier experiments (cited in this paragraph) and if necessary repeat the experiments when the specimens are subjected to biaxial "zig-zag" paths in *strain space*.

We now turn to the evolution of yield surfaces and the closely related study of strain-hardening behavior. Here more than in any other aspect of plasticity, simultaneous measurements in both strain and stress spaces are essential. With reference to the strain-hardening characterization defined in (4.17), it is clear that measurements of the increments of the normal outward displacement u_s of the yield surface in stress space and the normal outward displacement u_e of the yield surface in strain space enable one to directly calculate the ratio $\Delta u_s / \Delta u_e$; and this, in turn, permits measurement of the function $\Phi = \hat{f}/\hat{g}$. We recall in this connection that the quotient $\hat{f}/\hat{g}$, which is rate-independent is related to the ratio of the outward normal velocities of the yield surfaces (in stress and strain spaces) during plastic flow according to the formulae (see Casey and Naghdi 1983c, Eqs. (30)–(31)):

$$v_s = \frac{du_s}{dt} = \hat{f}\left(\frac{\partial f}{\partial \boldsymbol{S}} \cdot \frac{\partial f}{\partial \boldsymbol{S}}\right)^{-1/2}, \tag{7.1}$$

$$v_e = \frac{du_e}{dt} = \hat{g}\left(\frac{\partial g}{\partial \boldsymbol{E}} \cdot \frac{\partial g}{\partial \boldsymbol{E}}\right)^{-1/2} > 0, \tag{7.2}$$

so that

$$\frac{\hat{f}}{\hat{g}} = \frac{du_s}{du_e} \frac{\left(\dfrac{\partial f}{\partial \boldsymbol{S}} \cdot \dfrac{\partial f}{\partial \boldsymbol{S}}\right)^{1/2}}{\left(\dfrac{\partial g}{\partial \boldsymbol{E}} \cdot \dfrac{\partial g}{\partial \boldsymbol{E}}\right)^{1/2}}. \tag{7.3}$$

The formula (7.3) also requires knowledge of the ratio of the magnitudes of the normals $\partial f/\partial \boldsymbol{S}$ and $\partial g/\partial \boldsymbol{E}$. But, at least in simple cases, this ratio can be

readily determined from measurement of the elastic properties of the material represented by the fourth order tensor $\mathscr{L}$ in the expression (4.15) in subsection 4E.

To continue the discussion on the nature of the function Φ, we observe that this function depends on the variables $\mathscr{V}$ defined by $(4.7)_1$ and possibly on the shift tensor $\boldsymbol{\alpha}$ introduced in subsection 4H, or equivalently on the variables (5.2), i.e.,

$$\Phi = \Phi(\boldsymbol{E}, \mathscr{W}), \tag{7.4}$$

where $\mathscr{W}$ is defined by (5.4). Plots of the function Φ for various values of its arguments should provide substantial information regarding the strain-hardening characteristics of the material (see also Table 1 of subsection 4F)[71]. It is instructive at this point to consider a special choice for the yield functions g and f; and, then, in the presence of small deformation, to illustrate certain features of the function Φ. To this end consider experiments of the type reported by Lamba and Sidebottom (1978a,b) for two-dimensional strain cycling that can be sustained by a biaxial state of stress resulting from combined tension-compression and torsion of thin-walled specimens of OFHC copper[72]. Again, in the context of small deformation and utilizing generalized Hooke's law, we recall from (4.27) the expressions for the yield functions f and g in terms of the deviatoric and spherical parts of strain and stress and now write the function g given by $(4.27)_3$ in the form (for details see Naghdi and Nikkel 1986)[73]:

$$g = \tfrac{2}{3}E^2\left[e_{11} - \left(1 + \tfrac{3}{4}\frac{\bar{\alpha}}{E}\right)e^p_{11}\right]^2 + 8\mu^2\left[e_{12} - \left(1 + \frac{\bar{\alpha}}{4\mu}\right)e^p_{12}\right]^2 - \kappa, \tag{7.5}$$

where the coefficient $\bar{\alpha}$ is defined by

$$\bar{\alpha} = \bar{\alpha}(\kappa) = \frac{(\alpha_0 - \alpha_s)\kappa + \alpha_s\kappa_0 - \alpha_0\kappa_s}{\kappa_0 - \kappa_s}, \tag{7.6}$$

and in writing (7.5) the lateral strain components $e_{22} = e_{33}$ and $e^p_{22} = e^p_{33}$ have been eliminated from the yield function since they can be expressed in terms of e_{11} and e^p_{11}. Also, in (7.5)–(7.6), E is the Young's modulus of

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71 In this connection, mention should also be made of two interesting papers of Casey and Lin (1983, 1984), where strain-hardening topography of elastic-plastic materials is discussed.

72 The abbreviation OFHC stands for "oxygen-free high conductivity." The experimental data of Lamba and Sidebottom (1978a,b) were obtained during loading while the material was hardening in the sense summarized in Table 1 of subsection 4F.

73 Recall that in the context of a general theory summarized in section 5, the yield function g in strain space—apart from its dependence on $\boldsymbol{E}$, is also a function of the variables $\mathscr{W}$ defined by (6.4). As far as dependence on the variables $\mathscr{W}$ is concerned, a parallel statement holds for the yield function f in stress space. A special choice for the shift tensor would be the case in which $\boldsymbol{\alpha}$ is taken to be a function of $(\boldsymbol{E}_p, \kappa)$ and a more restrictive choice arises when $\boldsymbol{\alpha}$ is a scalar function of κ only as in $(4.27)_{1,2}$ and (7.5) with $\bar{\alpha}$ specified by (7.6).

elasticity, μ is the elastic shear modulus and κ_0 and κ_s are the values of κ at initiation of yield and at saturation, respectively. Moreover, $\bar{\mathfrak{z}}$ specified by (7.6) has the property that when $\kappa = \kappa_0$, $\bar{\mathfrak{z}}$ reduces to $\mathfrak{z}_0$ and when $\kappa = \kappa_s$, $\bar{\mathfrak{z}}$ reduces to $\mathfrak{z}_s$.

For the sake of clarity in what follows, it is helpful to note certain simplifications that occur in the representation of the function Φ for the special case in which the stress response does not depend on the second and third entries in the set of variables $(4.7)_{1,2}$. Then, remembering that generalized Hooke's law has already been used in the preceding paragraph, by an appeal to the work assumption of Naghdi and Trapp (discussed in subsection 5C), the right-hand side of (7.4) can be simplified considerably[74]. With $\mathfrak{z}$ chosen to be a scalar function of κ as in the expressions $(4.27)_{1,2}$ and (7.5), it follows that Φ depends only on the total strain $\boldsymbol{E}$, and the plastic strain $\boldsymbol{E}_p$, as well as κ. Moreover, during plastic deformation the set of variables $\mathscr{U} = (\boldsymbol{E}, \boldsymbol{E}_p, \kappa)$ must satisfy the yield condition $g = 0$ with g specified by (7.5). It turns out that for the particular case under discussion the dependence of Φ on the variables $\mathscr{U}$ can be expressed in the form[75]

$$\Phi = \hat{\Phi}\left[e^p_{11} - \left(1 + \tfrac{3}{4}\frac{\mathfrak{z}_0}{E}\right)^{-1} e_{11}, e^p_{12} - \left(1 + \frac{\mathfrak{z}_0}{4\mu}\right)^{-1} e_{12}, \kappa\right], \tag{7.7}$$

The last result must be solved simultaneously with the equation resulting from setting the right-hand side of (7.5) equal to zero, i.e.,

$$\begin{aligned} &\kappa = \text{The expression resulting from (7.5) by setting } g = 0, \\ &\kappa_0 \le \kappa \le \kappa_s. \end{aligned} \tag{7.8}$$

In order to obtain a suitable plot of the function Φ, use was made of the experimental data of Lamba and Sidebottom (1978a,b) in two-dimensional strain cycling. In the course of identifying values for the various coefficients from this data, it was found that the values of $\mathfrak{z}_0$ and $\mathfrak{z}_s$ differed only by less than 0.3 percent and hence by neglecting this difference one could assume the two coefficients to have the same value in this calculation, i.e., $\mathfrak{z}_0 = \mathfrak{z}_s$. The values of all material constants used in the calculation are given in Eqs. (4.1) of Naghdi and Nikkel (1986). A plot of the variation of Φ with plastic strains (e^p_{11}, e^p_{12}) for fixed values of (e_{11}, e_{12}) will represent a surface in a 3-space with coordinates $(\Phi, e^p_{11}, e^p_{12})$. For definiteness, we specify $e_{11} = e_{12} = 0$ and then calculate the value of Φ for each pair of (e^p_{11}, e^p_{12}) that

[74] For details of the simplified form of the function Φ which (instead of $\hat{f}$ $\hat{g}$) depends only on the partial derivatives of f and g, see Naghdi and Nikkel (1984, Eq. (7) and 1986, Eq. (2.10)). This representation of Φ is based on earlier developments in the papers of Casey and Naghdi (1981a, between Eqs (36)-(42) and 1984a, Eq. (4.5)).

[75] An explicit form for the function $\hat{\Phi}$ is not recorded here, but can be readily calculated from the detailed development given in Naghdi and Nikkel (1986).

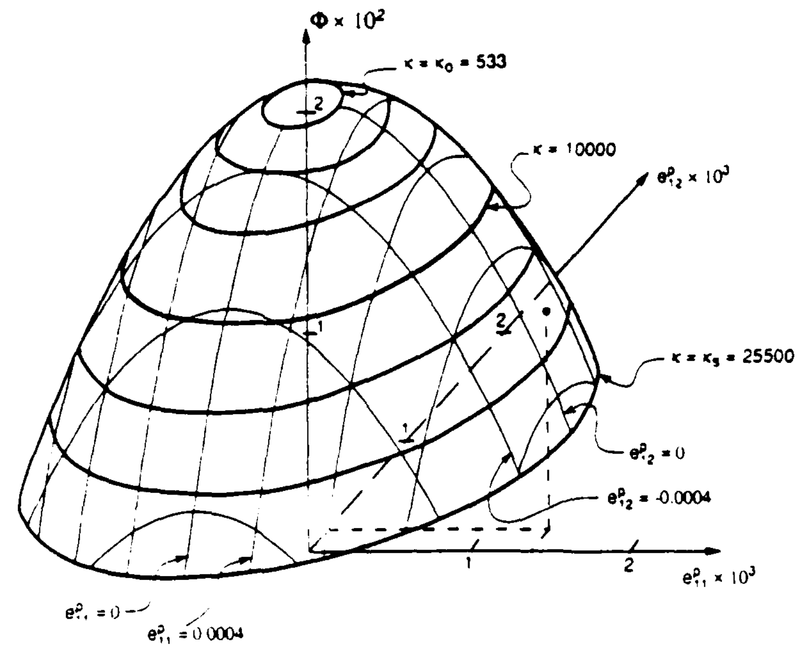

Figure 5
A geometrical representation of the strain-hardening function Φ as a surface exhibiting its dependence on the plastic strain components e^p_{11} and e^p_{12}, plotted for fixed values of the total strains (taken in this figure to be $e_{11} = e_{12} = 0$) for OFHC copper tested in strain cycling experiments of Lamba and Sidebottom (1978a,b). The heavier closed curves on the surface are curves of constant κ bounded by the values κ_0 at initial yield and κ_s at saturation; the values of other κ-curves shown are listed in Table 2. For different fixed values of e_{11} and e_{12}, the surface does not change its shape but merely translates parallel to the plane of (e^p_{11}, e^p_{12}).

also satisfy (7.8) for each[76] κ. A plot of a surface of the type just described (a semi-ellipsoidal surface) was included in a paper of Naghdi and Nikkel (1986) and is reproduced here as[77] Fig. 5. The lighter curved lines in Fig. 5, which collectively represent the semi-ellipsoidal surface, are lines of constant plastic strains (e^p_{11}, e^p_{12}) drawn at intervals of 0.0004 (only a few of these are shown). The heavier curves on the surface are curves of constant κ bounded by the values of $\kappa_0 = 533\ (\mathrm{MPa})^2$ and $\kappa_s = 25500\ (\mathrm{MPa})^2$. Only a few of these κ-curves are shown in Fig. 5; see also Table 2 which provides a list of values of Φ for corresponding values of κ. It should be noted that for a fixed κ in the range $\kappa_0 \le \kappa \le \kappa_s$, say for $\kappa = 10000\,(\mathrm{MPa})^2$, any point on the κ-curve corresponds to a point on the yield surface $g = 0$ with the same value of κ.

[76] It should be emphasized that in such calculations the choice of (e_{11}, e_{12}) and (e^p_{11}, e^p_{12}) cannot be arbitrary and must be such that they satisfy the yield condition $g = 0$ with g specified by (7.5).

[77] The present Fig. 5 is a different version of the previously published figure. I thank D. Nikkel for this revised and perhaps more illuminating plot of the data originally presented in Naghdi and Nikkel (1986, Fig. 4).

Table 2
Representative values of κ (between κ_0 and κ_s) and the corresponding values of the (dimensionless) strain-hardening measure Φ. The heavier curves of constant κ shown on the surface in Fig. 5 correspond to the values of κ in Table ~~1~~ 2

κ in $(\mathrm{MPa})^2$	$\Phi \times 10^2$
533	2.04
2000	1.95
5000	1.78
10000	1.48
15000	1.18
20000	0.878
25500	0.545

It is clear from the arguments of the function $\hat{\Phi}$ in (7.4) that the single calculation on which Fig. 5 is based provides all the relevant information on the variation of Φ for strain-hardening characterization; and that, for any other specified values of e_{11} and e_{12}, the surface plotted in Fig. 5 will not change in shape but will simply translate parallel to the $e^p_{11} - e^p_{12}$ plane by the constant amounts

$$\left(1 + \tfrac{3}{4}\frac{\varkappa_0}{E}\right)^{-1} e_{11}, \left(1 + \frac{\varkappa_0}{4\mu}\right)^{-1} e_{12} \tag{7.9}$$

in the e^p_{11} and e^p_{12} directions, respectively. Some additional features of the surface plotted in Fig. 5 should be noted: (1) it is symmetric with respect to the $e^p_{11} - \Phi$ and $e^p_{12} - \Phi$ planes in the sense that the values of Φ will remain unchanged if e^p_{11} is changed to $-e^p_{11}$ and if e^p_{12} is changed to $-e^p_{12}$; (2) the maximum value of Φ occurs at the lower value of $\kappa = \kappa_0$ near the apex of the surface (the closed curve at the top identified as $\kappa = \kappa_0 = 533$) and decreases to its minimum at the higher value of $\kappa = \kappa_s$ on the bottom of the semi-ellipsoidal surface (the closed curve identified as $\kappa = \kappa_s = 25500$).

The foregoing discussions in this subsection have emphasized topics in experimental plasticity involving ductile metals, even though many of the suggested experiments could also be carried out for nonmetallic materials. One particular class of materials to which we expect a general theory of elastic-plastic materials to be applicable is represented by geomaterials such as rock, soil and concrete. In an interesting series of experiments pertaining to softening of rock, Wawersik and Fairhurst (1970) and Hudson, Brown and Fairhurst (1972) have reported their test data in small deformation of unconfined compression of cylindrical specimens of different marbles[78] in

[78] Six rock-type materials, including Tennessee marble, Charcoal Gray granite and Indiana limestone, were used by Wawersik and Fairhurst (1970) while the specimens in the study of Hudson *et al.* (1972) were made of Georgia Cherokee marble.

order to establish whether the strain softening is entirely a real material property or is a result of specimen size, shape or even testing techniques. In a related informative review article on the subject, Read and Hegemier (1984) have analyzed these experimental results and have noted the fact that the softening behavior observed by Hudson *et al.* (1972) and others is partly due to degradation of the material and does not necessarily represent softening of the material itself. To see this, let t_{11} be the axial component of the stress so that $t_{11} = (F/a)$, where F is the applied compressive force and A is the cross-sectional area of the specimen undergoing small deformation. In an actual test the load-bearing cross-sectional area of the specimen diminishes due to the slabbing (i.e., axial fracturing) of the material at or near the lateral free surface and hence the actual cross-sectional area of the specimen, say A', will be less than A and hence the actual axial stress (instead of t_{11}) will be $t'_{11} = (F/A')$. It then follows that $t'_{11} = (A/A')t_{11}$, indicating a larger ultimate stress. Thus, unless the actual load-bearing area A' is used in the calculation of the stress, the stress-strain curve will appear increasingly lower than it should be.

All of the experimental results mentioned or discussed so far in this subsection have been obtained in the context of small deformation. In the future, such experiments should be repeated in the presence of large strains. As may be surmised, fundamental experiments for finitely deforming elastic-plastic materials are rare. For example, in an interesting series of experiments by Armstrong, Hockett and Sherby (1982), 1100 aluminum cubes at room temperature were subjected to multidirectional compression in order to study the material behavior at large plastic strain. In another recent and noteworthy paper, Bell (1989) has reported experimental data from a large number of thin-walled tubular specimens of different materials, including metal alloys, that have been twisted and extended well into the plastic range (to as much as 360° in twist and in 30% in axial direction). These large strains were obtained by subjecting the specimens to combined tension, torsion and internal pressure, and the tests were conducted in such a manner that the originally circular cylindrical specimens remained so even at large deformation. A particularly striking result of Bell's (1989) experimental data is the demonstration that the twisting and extension to very large strains is accompanied by only a small change in the rotation tensor[79]. Mention should also be made of other experimental results discussed by Bell (1986)[80], which include some finite strain measurements and where a list of

[79] Recall that (by the polar decomposition theorem) the deformation gradient F can be expressed as a product of two second order tensors one being a symmetric positive definite tensor representing stretch and the other being an orthogonal tensor representing rotation.

[80] In this paper Bell also indicates an observed parabolic relationship between a scalar "effective" stress measure T and a scalar "effective" strain measure Γ (see Bell 1986, Eqs. (4) and (5)). The connection between this relation and the constitutive ingredients of a general theory of elastic-plastic materials is unclear at this time.

related experimental papers by him and his co-workers can be found. Clearly the conduct of experimental studies of this kind should be more widespread and should also be extended to include data from unloading curves, as well as other experimental features in finite plasticity discussed in this subsection.

7B. *Computational aspects*

In the computational literature, rather than incrementing the stress as an independent variable, frequently strain increments are used to calculate stress increments. While such a procedure is in harmony with the strain-space formulation of plasticity, it is also important to ensure that any special loading criteria used in the computation are compatible with the loading criteria of the strain-space formulation defined by (4.16). Moreover, it is also necessary to include the yield function in stress space along with its loading conditions summarized in the paragraph following (4.16).

Apart from the obvious merits of the current computational capabilities in facilitating calculation of specific results and theoretical predictions, numerical simulations can be used to great advantage in supplementing the experimental results by identifying and delineating new directions for further theoretical developments. In fact, such numerical simulations could be undertaken in parallel with some of the experiments suggested in subsection 7A and in some instances could have a clarifying influence in situations for which experimentation is rather difficult if not impossible. For example, by numerical simulations, it should be possible to construct a plot or plots of the strain-hardening characterization represented by the function Φ for more general situations than that illustrated in Fig. 5 and discussed in the previous subsection. Indeed, such numerically obtained results would complement the direct experimental measurements of Φ suggested in subsection 7A. Similarly, numerical simulation in both strain and stress spaces could shed light on the geometric evolution of yield surfaces during plastic flow.

8. Microstructural effects and crystal plasticity

We discuss here aspects of microstructural effects in metallic materials. The term *microstructure* refers to the mechanical properties of metals which occur on a microscale, i.e., a scale lower (or finer) than the macroscale[81]. A descriptive account of the subject, which also includes some historical background, can be found in an easily readable article by Cottrell (1967).

[81] We exclude here other properties of metals, such as optical and electromagnetical, which occur on a scale still lower (or finer) than the microscopic scale.

For more general background information on dislocation theory and plastic flow in crystals, we refer the reader to the books by Cottrell (1953), Nabarro (1987), Hirth and Lothe (1982) and Hertzberg (1983).

What is commonly referred to as *physical plasticity* attempts to incorporate into (macroscopic) continuum mechanics certain well-known experimental observations at the microscopic level. Clearly, once a fairly general structure of a continuum theory is available, it should possess sufficient flexibility to also accommodate microstructural effects. In other words, the point of view taken here is that (macroscopic) continuum theories should have a fairly broad base and a general outlook that allow for the additional microscopic ingredients if desired. Such a broad continuum theory was presented in previous sections of this review article (such as sections 5 and 6); and while this theory does not explicitly account for microstructural effects, it allows for possible inclusion of such microstructural notions. These additional ingredients can be incorporated into the continuum theory in a variety of ways. In the present section, we focus attention on a widely accepted framework for including crystallographic features in a macroscopic theory of finitely deforming elastic-plastic materials. This line of research actually predates not only the modern approach to finite plasticity but also a good deal of the classical theory of elastic-plastic materials with small deformation. Pioneering work on the subject was laid by Orowan (1934), Polanyi (1934) and Taylor (1934), and subsequently was extended by Hill (1966). More recently, the subject has been pursued in somewhat different directions by Mandel (1973, 1981, 1982), Havner and Shalaby (1977), Asaro and Rice (1977), Weng (1980), Havner (1982), Hill and Havner (1982), Iwakuma and Nemat-Nasser (1984), and has been reviewed extensively by Asaro (1983a,b).

In order to make the context of this section accessible to those readers who may not be intimately familiar with micromechanics of crystalline materials, i.e., single crystals or polycrystalline grains, it is perhaps desirable to include here some additional background information. Briefly, slip is caused by glide of dislocations across a slip plane one lattice spacing at a time. A relatively simple and informative description of slip in crystalline materials can be found in Nabarro (1987, Ch. I) who describes various types of dislocations and other developments on the subject (Nabarro 1987, Ch. I, especially pp. 5–8). Central to the current understanding of the crystallographic nature of slip is the work of Taylor and Elam (1923, 1925) who studied in considerable detail plastic deformation of aluminum single crystals. In these papers, which bear significantly on the physics of plastic deformation of the crystalline structure of metals, Taylor and Elam (1923, 1925) provided an appropriate interpretation of the kinematics of deformation in terms of crystallographic structure. After identifying the slip planes and the slip directions and observing that slip occurs preferentially, i.e., that

some (but not all) slip directions are activated as a result of dislocation motion, they go on to identify the component of the shear stress in the slip plane and in the direction of slip, i.e., the *resolved* shear stress on the slip plane. Moreover, they noted that plastic yielding occurs in a single crystal in accordance with the Schmid law (Schmid 1924), i.e., when the resolved shear stress on the slip plane reaches a critical value. Remembering the definition of slip (stated earlier in this paragraph), we now note that the terminology of the slip system or "active" slip system used in much of the current literature refers to the combination of slip planes and slip directions.

The majority of the recent contributors to the (macroscopic) continuum theory which takes into account microstructural effects appear to subscribe to the hypothesis that material moves across the crystal lattice structure as a result of dislocation, while the lattice itself undergoes only elastic deformation (see Asaro 1983a, p. 36). Evidently these notions, suggested by experimental observations, form the basis of the currently accepted point of view for associating certain features with the macroscopic *model* upon which a continuum theory can be constructed. Here, based on these observations, we regard the deformation process from a reference configuration κ_0 to the current configuration κ at time t to be broken into three parts[82]: (1) plastic shearing of the material which does not alter the lattice structure, (2) a local rotation of both the material and the lattice, followed by (3) an elastic deformation in which both the material and the lattice participate.

In the remainder of this section we provide separately a discussion of kinematics of deformation and the nature of constitutive laws in a (macroscopic) continuum theory for a single slip system, as well as an assessment of existing constitutive developments in the continuum theory of elastic-plastic crystals. Our discussion will focus only on the rate-independent aspects of elastic-plastic crystals and this is in line with most of the existing developments aimed at formulating a continuum theory of plasticity which includes microstructural effects.

8A. *Preliminary analysis of a single slip system*

The purpose of this subsection is to provide in the context of (macroscopic) continuum mechanics a preliminary analysis for elastic-plastic deformation of a single crystal. As in most of the recent literature cited earlier in this section, our development will be based on the basic features of a simple slip system and the related assumptions [described in the fourth paragraph of this section] for incorporating microstructural effects into the macroscopic theory. These features are schematically depicted in Fig. 6 with the use of the multiplicative decomposition of the type (4.1) associated with an

[82] The detailed nature of these assumptions are spelled out below in subsection 8A.

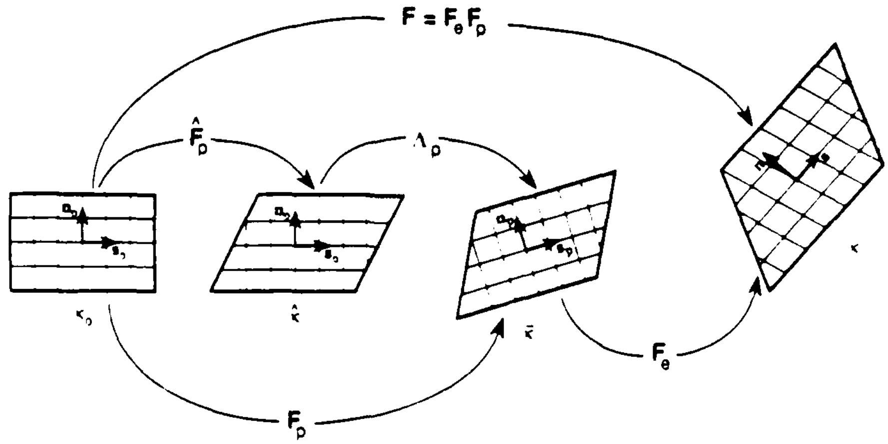

Figure 6
A schematic sketch representing (on the macroscopic scale) elastic-plastic deformation in a crystalline medium resulting from the motion of a single slip plane from its reference configuration $\boldsymbol{\kappa}_0$ to the current configuration $\boldsymbol{\kappa}$. Also shown is an intermediate configuration $\bar{\boldsymbol{\kappa}}$ and the associated multiplicative decomposition of the deformation gradient $\boldsymbol{F}$ into the factors representing the "elastic" part $\boldsymbol{F}_e$ and the "plastic" part $\boldsymbol{F}_p$. The motion between $\boldsymbol{\kappa}_0$ and $\bar{\boldsymbol{\kappa}}$ is composed of two parts, namely $\hat{\boldsymbol{F}}_p$ which has no effect on the reference orthonormal vectors $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ and the rotation $\boldsymbol{\Lambda}_p$ which only rotates $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ into $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ in the configuration $\bar{\boldsymbol{\kappa}}$. Only the "elastic" part $\boldsymbol{F}_e$ of the deformation gradient acts on the unit vectors $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ which become $(\boldsymbol{s}, \boldsymbol{n})$ in $\boldsymbol{\kappa}$.

intermediate stress-free configuration[83]. It should be noted here that the details of Fig. 6 differ from the corresponding schematic diagram in Asaro (1983a, p. 38, Fig. 29), but are compatible with the analysis given below in this subsection. Asaro's deformation process does not include the part between $\hat{\boldsymbol{\kappa}}$ and $\bar{\boldsymbol{\kappa}}$ in Fig. 6, i.e., the local rotation $\boldsymbol{\Lambda}_p$ which is not a *rigid body* rotation.

To continue the discussion, let $\boldsymbol{X}$, $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ be the position vectors in the continuum theory corresponding to a single slip system in the reference configuration $\boldsymbol{\kappa}_0$, an intermediate stress-free configuration $\bar{\boldsymbol{\kappa}}$ and the current configuration $\boldsymbol{\kappa}$ at time t. Recalling the definition $(2.1)_1$ for the deformation gradient $\boldsymbol{F}$ relative to the reference position, we may write

$$d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}, \tag{8.1}$$

which is a linear transformation taking a line element $d\boldsymbol{X}$ into $d\boldsymbol{x}$. The use of the multiplicative decomposition (4.1) then permits replacing (8.1) with

$$d\boldsymbol{x} = \boldsymbol{F}_e\, d\bar{\boldsymbol{x}}, \qquad d\bar{\boldsymbol{x}} = \boldsymbol{F}_p\, d\boldsymbol{X}. \tag{8.2}$$

[83] Our main reason for adopting such a decomposition in this subsection is to provide an easy comparison with the existing literature (see Asaro 1983a), most of which utilize a decomposition of the form (4.1). We note, however, that such decompositions are too restrictive and are subject to the same type of criticism already discussed in subsection 4A, especially in the paragraph following (4.3).

Again, with reference to Fig. 6, let the vector fields $(\boldsymbol{s}, \boldsymbol{n})$—not necessarily unit vectors—represent, respectively, the slip direction and the normal to the slip plane in the current configuration $\boldsymbol{\kappa}$. Further, let $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ and $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ be the values of $(\boldsymbol{s}, \boldsymbol{n})$ in the intermediate configuration $\bar{\boldsymbol{\kappa}}$ and the reference configuration $\boldsymbol{\kappa}_0$. The vectors $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ are assumed to be orthonormal and, as will be seen presently, $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ are also orthonormal.

Remembering the background information for the single slip system shown in Fig. 6, the deformation from $\boldsymbol{\kappa}_0$ to $\boldsymbol{\kappa}$ consists of three parts. These parts consist of a deformation measure representing plastic shearing of the material and characterized by a second order tensor $\hat{\boldsymbol{F}}_p$ from $\boldsymbol{\kappa}_0$ to a configuration $\hat{\boldsymbol{\kappa}}$, followed by a local rotation of the slip system characterized by a second order orthogonal tensor Λ_p from $\hat{\boldsymbol{\kappa}}$ to the intermediate stress-free configuration $\bar{\boldsymbol{\kappa}}$, and a subsequent elastic deformation measure $\boldsymbol{F}_e$ from $\bar{\boldsymbol{\kappa}}$ to κ. Consistent with these assumptions (see also Fig. 6), it is seen that $\boldsymbol{F}_p$ can be related to $\hat{\boldsymbol{F}}_p$ and Λ_p in the form

$$\boldsymbol{F}_p = \Lambda_p \hat{\boldsymbol{F}}_p, \qquad \hat{\boldsymbol{F}}_p = \boldsymbol{I} + \gamma \boldsymbol{s}_0 \otimes \boldsymbol{n}_0, \tag{8.3}$$

where the scalar-valued function γ in $(8.3)_2$ represents a measure of shear strain relative to the undeformed lattice in $\boldsymbol{\kappa}_0$. Clearly, in view of $(8.3)_1$, the multiplicative decomposition (4.1) can now be expressed in the form[84]

$$\boldsymbol{F} = \boldsymbol{F}_e \Lambda_p \hat{\boldsymbol{F}}_p. \tag{8.4}$$

In the first part of the deformation process discussed above the depicted in Fig. 6, $\hat{\boldsymbol{F}}_p$ acts on the material line $(8.2)_2$ but not on the orthonormal vectors $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ representing the lattice structure. Thus, in the course of motion from $\boldsymbol{\kappa}_0$ to $\bar{\boldsymbol{\kappa}}$, the unit vectors $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ remain unaffected by plastic shearing from $\boldsymbol{\kappa}_0$ to $\hat{\boldsymbol{\kappa}}$ and only rotate from $\hat{\boldsymbol{\kappa}}$ to $\bar{\boldsymbol{\kappa}}$ so that the unit vectors $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ are also orthonormal. It then follows that the unit vectors $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ in $\bar{\boldsymbol{\kappa}}$ are related to $(\boldsymbol{s}_0, \boldsymbol{n}_0)$ in $\hat{\boldsymbol{\kappa}}$ by the transformations

$$\boldsymbol{s}_p = \Lambda_p \boldsymbol{s}_0, \qquad \boldsymbol{n}_p = \Lambda_p \boldsymbol{n}_0. \tag{8.5}$$

It should be emphasized that the decomposition $(8.3)_1$ necessarily implies consideration of another configuration $\hat{\boldsymbol{\kappa}}$ as indicated in Fig. 6 such that the deformation between $\boldsymbol{\kappa}_0$ and $\hat{\boldsymbol{\kappa}}$ involves the plastic deformation $\hat{\boldsymbol{F}}_p$ followed by a pure rotation Λ_p between $\hat{\boldsymbol{\kappa}}$ and $\bar{\boldsymbol{\kappa}}$.

So far as the kinematical aspect of the slip system is concerned, it remains to indicate the relationship between the vector fields $(\boldsymbol{s}, \boldsymbol{n})$ in $\boldsymbol{\kappa}$ and the orthonormal set $(\boldsymbol{s}_p, \boldsymbol{n}_p)$ in $\bar{\boldsymbol{\kappa}}$. Keeping in mind that by the nature of our basic assumptions only $\boldsymbol{F}_e$ acts on the lattice structure in $\bar{\boldsymbol{\kappa}}$, it follows that

$$\boldsymbol{s} = \boldsymbol{F}_e \boldsymbol{s}_p, \qquad \boldsymbol{n} = (\det \boldsymbol{F}_e)(\boldsymbol{F}_e^{-1})^T \boldsymbol{n}_p. \tag{8.6}$$

[84] Comparison of (8.4) with $\hat{\boldsymbol{F}}_p$ given by $(8.3)_2$ and (4.1) easily reveals the influence of microstructural effects on the basis of simple slip system described in this subsection and Fig. 6.

The first of (8.6) readily follows from the fact that only $\boldsymbol{F}_e$ acts on $\boldsymbol{s}_p$, but the second of (8.6) is less obvious and can be verified as follows: Let the scalars da and $d\bar{a}$ denote, respectively, elements of surface area in the configurations $\bar{\kappa}$ and κ. Then, recalling a well-known transformation relation between the area elements in $\bar{\kappa}$ and κ involving $\boldsymbol{n}_p d\bar{a} = d\bar{\boldsymbol{a}}$ and $d\boldsymbol{a}$, respectively, we have

$$d\boldsymbol{a} = (\det \boldsymbol{F}_e)(\boldsymbol{F}_e^{-1})^T \boldsymbol{n}_p \, d\bar{a},$$

from which after dividing by the scalar $d\bar{a}$ follows the result $(8.6)_2$.

We now turn our attention to the invariance properties of the various quantities utilized between (8.2)–(8.6) under s.r.b.m.[85] We recall from subsection 4C that under s.r.b.m. the second order tensor functions $\boldsymbol{F}$, $\boldsymbol{F}_e$, $\boldsymbol{F}_p$ transform according to $(4.5)_{1,2,3}$, where $\boldsymbol{Q}(t)$ and $\bar{\boldsymbol{Q}}(t)$ are two different proper orthogonal tensor functions of time, each representing a *rigid body* rotation. Then, by standard arguments in continuum mechanics, we have the following transformations

$$\begin{aligned} s_p^+ &= \bar{Q} s_p, \qquad n_p^+ = \bar{Q} n_p, \\ s^+ &= Q s, \qquad \boldsymbol{n}^+ = Q \boldsymbol{n}, \end{aligned} \tag{8.7}$$

from which and with the use of $(8.5)_2$ and $(8.7)_2$ we also have the result

$$A_p^+ = \bar{Q} A_p. \tag{8.8}$$

Moreover, it follows from $(4.5)_3$, $(8.3)_1$ and (8.8) that under s.r.b.m. $\hat{\boldsymbol{F}}_p$ transforms by

$$\hat{\boldsymbol{F}}_p^+ = \hat{\boldsymbol{F}}_p \tag{8.9}$$

for all orthogonal tensors $\boldsymbol{Q}(t)$. The results (8.8) and (8.9) were obtained previously by *Casey (1987) in his discussion* of Dashner's (1986) paper concerning the role of invariance requirements under s.r.b.m.[86]

The foregoing preliminary analysis, mainly kinematical, is based on a simple *model* of a single slip system. A more general analysis of microstructural effects should include multislip systems permitting several slip systems to be simultaneously active. Such a scheme could be effected by expressing the constitutive equations for the macroscopic variables such as the rate of plastic strain $\dot{\boldsymbol{E}}_p$ as a series sum over active slip systems $(\boldsymbol{s}^{(\alpha)}, \boldsymbol{n}^{(\alpha)})$ in the form (compare with Eq. (3.10) in Asaro 1983a):

$$\text{Rate of plastic strain} = \sum_{\alpha=1}^{n} \left\{ \begin{array}{l} \text{the constitutive} \\ \text{ingredients based on} \\ \text{multislip systems} \end{array} \right\} \boldsymbol{s}^{(\alpha)} \otimes \boldsymbol{n}^{(\alpha)}. \tag{8.10}$$

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[85] The abbreviation s.r.b.m. refers to superposed rigid body motions introduced in the opening paragraph of section 4. Also, as stated previously in section 2, for all quantities associated with the configuration κ^+ (as a consequence of s.r.b.m. from κ) we use the same symbol but with an attached plus "+" sign.

[86] In this connection see the remarks in the last paragraph of subsection 4C.

In (8.10) the series is summed over the active slip systems and the quantity in the bracket { } on the right-hand side is intended to represent an appropriate measure of deformation on the microscopic level relative to the undeformed lattice such as the rate of plastic shearing $\dot{\gamma}^{(\alpha)}$, where the scalar $\gamma^{(\alpha)}$ represents plastic shearing of the αth constitutive ingredient measured relative to the undeformed lattice in κ_0.

A complete theory in the context of crystal plasticity must include a detailed consideration of various aspects of constitutive equations and their ingredients along the lines discussed in subsections 4A to 4H of section 4 but suitably generalized to include microstructural effects. However, at the present time, the development of such constitutive equations are in their primitive stages and have not been fully understood even on the basis of a simple model of the type discussed in this subsection.

A further comment seems to be desirable regarding the possibility of introducing additional structure into the (macroscopic) theory in order to represent the microstructural effects. For this purpose, we may admit *directors* as additional independent vector fields, as has been intimated sometimes in the literature. With reference to the single slip system discussed in this subsection and shown in Fig. 6, this would require two directors to replace the vector fields $(\boldsymbol{s}, \boldsymbol{n})$ in the configuration[87] κ. The directors, which must satisfy appropriate invariance requirements under s.r.b.m., could be so constrained that their behavior would be compatible with those of the vector fields $(\boldsymbol{s}, \boldsymbol{n})$ of the single slip system throughout the deformation process between the configuration κ_0 and κ. But the use of the director fields could also capture other features at the microscopic level; and, in any case, the directors must satisfy appropriate dynamical equations supplementary to the equations of motion (3.3) or (3.2)[88]. Moreover, the constitutive equations must now include the director fields and possibly their gradients as additional constitutive ingredients. Such a development and its interpretation is not available at present.

8B. *Nature of progress in crystal plasticity*

We discuss in this subsection the current state of theories of finitely deforming elastic-plastic materials which incorporate microstructural effects. The direction and scope of research in this area has been markedly different from that pursued during the two decades between 1945–65. The development of the subject has advanced considerably during the period 1966–84, particularly from the point of view of utilizing and incorporating important

[87] Of course, additional directors are needed in the case of multislip systems.

[88] Background information for theories that utilize directors appropriate for directed media (also called Cosserat continua) can be found in Truesdell and Toupin (1960) and Naghdi (1972).

observations from fundamental experiments using metal crystals. Indeed, beginning with Hill's (1966) paper, gradually the development of the subject has progressed to its current level as reflected by the contents of some of the papers cited in the second paragraph of this section, especially the informative review article of Asaro (1983a). Nevertheless, for the convenience of future researchers in this area, it may serve a useful purpose to call attention to some shortcomings in the existing formulations of crystal plasticity. With this in mind, we first note that the constitutive equations in all of the recent papers cited earlier in this section (second paragraph) utlilize a stress-based formulation, rather than a strain-based formulation in the sense discussed in sections 4–6; and this aspect of all of these papers has the same anomalous difficulties as do the stress-based theories that do not explicitly include microstructural effects. Apart from this, we now proceed to discuss the nature of a few representative papers beginning with Mandel's (1973) paper which appears to be one of the early attempts for introducing an additional structure into the basic framework of the theory of elastic-plastic materials.

As stated in subsection 6A (third paragraph), for the purpose of incorporating the effect of orientation of a microelement into the macroscopic theory, Mandel (1973) introduces an "orientation variable" which he calls a director triad. But his choice for the representation of an "orientation variable" in terms of an *orthonormal* set of directors is much too restrictive, as was already noted at the end of the third paragraph of subsection 6A. Similar mention of a director triad or directors is made in other papers (Mandel 1981 and 1982; Havner 1982, Sec. 4.1, p. 281), without any clear explication of the properties of such vector fields (e.g., invariance under s.r.b.m.) or the dynamical equations that they must satisfy. Moreover, in all of the papers already cited in this paragraph, use is made of a multiplicative decomposition of the type (4.1) and subsequent use of additive decomposition of the form (4.21) and a constitutive equation of the type (4.22).

Asaro's (1983a, p. 36) interpretation of his own hypothesis (also referred to in the fourth paragraph of this section) for describing the deformation process of a single slip system in terms of the multiplicative decomposition of the type (4.1) differs from that described in subsection 8A and shown in Fig. 6 (compare with Fig. 29, p. 38 of Asaro 1983a). In terms of the notation used here and as was noted earlier (first paragraph of subsection 8A), his deformation process from κ_0 to κ does not include the part represented by the local rotation Λ_p in Fig. 6. However, the right-hand side of his $\boldsymbol{F}_p$ is the same as the right-hand side of the expression for $\hat{\boldsymbol{F}}_p$ defined by $(8.3)_2$. One consequence of this flaw in Asaro's analysis is the fact that his ingredients (Asaro 1983a, pp. 37–39) are not properly invariant under s.r.b.m. and this, in turn, influences his development pertaining to his identification of his macroscopic variable for the rate of plastic deformation

in terms of rate of shearing in the slip directions of his multislip systems. As in many other papers discussed earlier, he also employs a variant of additive decompositions of the forms (4.21) and a constitutive equation of the type (4.22) (see Asaro 1983a, pp. 38–45). Again, it should be noted that a constitutive equation of this kind for the rate of plastic deformation falls in the same category as that criticized in subsection 4G. A further discussion of his (Asaro 1983a,b) constitutive developments includes the adaptation of the Schmid rule to characterize the yield condition, a set of constitutive equations for the flow rule and strain-hardening which incorporate microstructural effects. For details of these developments we refer the reader to Asaro (1983a, pp. 38–53). Asaro (1983a,b) also considers some constitutive restrictions by an appeal to the type of normality condition proposed by Hill and Rice (1973), but this aspect of his analysis is subject to the same criticism already noted in section 5 (last paragraph) and need not be repeated here.

In light of the above remarks, it is apparent that significant progress has been made in the area of micromechanics of crystal plasticity during the past decade. It should be evident, however, that in order to incorporate microstructural effects into theories of the type discussed in sections 4–6, a great deal of work is needed in both theoretical and experimental aspects of crystal plasticity. Given this state of the subject, one may anticipate that fundamental advances in this area will be forthcoming in the years that lie ahead.

Acknowledgment

The results reported here were obtained in the course of research supported by the Solid Mechanics Program of the U.S. Office of Naval Research under Contract N00014-84-K-0264, Work Unit 4324-436 with the University of California, Berkeley.

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Abstract

The object of this paper is to provide a critical review of the current state of plasticity in the presence of finite deformation. In view of the controversy regarding a number of fundamental issues between several existing schools of plasticity, the areas of agreement are described separately from those of disagreement. Attention is mainly focussed on the purely mechanical, rate-independent, theory of elastic-plastic materials, although closely related topics such as rate-dependent behavior, thermal effects, experimental and computational aspects, microstructural effects and crystal plasticity are also discussed and potentially fruitful directions are identified.

A substantial portion of this review is devoted to the area of disagreement that covers a detailed presentation of argument(s), both *pro* and *con*, for all of the basic constitutive ingredients of the rate-independent theory such as the primitive notion or definition of plastic strain, the structure of the constitutive equation for the stress response, the yield function, the loading criteria and the flow and the hardening rules. The majority of current research in finite plasticity theory, as with its infinitesimal counterpart, still utilizes a (classical) stress-based approach which inherently possesses some shortcomings for the characterization of elastic-plastic materials. These and other anomalous behavior of a stress-based formulation are contrasted with the more recent strain-based formulation of finite plasticity. A number of important features and theoretical advantages of the latter formulation, along with its computational potential and experimental interpretation, are discussed separately.

(Received: December 28, 1989)